AN UPDATE ON COMBINATORIAL METHOD FOR AGGREGATION OF EXPERT JUDGMENTS IN AHP

ABSTRACT

The paper covers the recent research of two versions of combinatorial method of deriving individual and group priorities from pair-wise comparison matrices (PCM). The method is based on enumeration of all possible basic sets of pair-wise comparisons (PC) from a given PCM and calculation of average priorities across all of them. In our paper we briefly describe the modified version of the method, which, in contrast to ordinary version, considers consistency, compatibility, and detail of expert data, and (if necessary) provides opportunities for expert data quality improvement.

Keywords: AHP/ANP, expert judgment, priority, pair-wise comparison matrix, spanning tree, consistency, compatibility, expert estimation scale.

1. Introduction

Calculation of priorities based on individual and group expert judgments is an essential step of AHP/ANP algorithm. Aggregate priorities, calculated using different methods are, in the general case, different from each other. The question "which aggregation method works better?" (eigenvector method, geometric mean, logarithmic least squares, combinatorial method, other) is still open to discussion. Combinatorial method (the subject of this paper) is based on enumeration of all possible basic sets of PCs from a given PCM (not just columns or rows). It utilizes expert data redundancy most thoroughly and allows us to check consistency, compatibility, and completeness of this data prior to aggregation. It also allows us to easily detect the most inconsistent/incompatible elements of initial PCM and, if necessary, ask respective experts to reconsider their judgments. The method is often called "enumeration of all spanning trees". Every basic PC set can be represented by a spanning tree. Its vertices represent compared objects, while edges represent respective PCs of the objects. In the current paper we will revisit the two conceptual versions of the method and demonstrate the advantage of the modified one over the ordinary one.

2. Literature Review

The idea to combinatorically enumerate all spanning trees, representing all PC sets, was suggested in 2000 and later elaborated by (Tsyganok et al. 2018). In (Kadenko et al, 2019) efficiency of combinatorial method was empirically shown. (Lundy et al, 2017) proved the equivalence of combinatorial and row geometric mean aggregation methods. (Bozoki & Tsyganok, 2019) proved the equivalence of logarithmic least squares and combinatorial method. (Lundy et al, 2017) and (Bozoki & Tsyganok 2019) focus on the ordinary individual method, where all spanning trees have the same weights. In (Kadenko et al, 2019) we show that "ordinary" and "weighted" versions of the method are not equivalent. We assume, that spanning trees (and respective basic PC sets) have different weights, depending on consistency, compatibility, completeness, and detail of initial expert data. Rating of basic PC sets also allows us to organize feedback with experts (revise specific judgments). Authors of the listed publications consider individual/group judgments, complete/incomplete PCM.

3. Hypotheses/Objectives

The purpose of this study is to show that in spite of recently obtained results ((Lundy et al, 2017), (Bozoki & Tsyganok, 2019)), it still makes sense to use and further improve the modified combinatorial method for aggregation of expert judgments, because it is conceptually different from and has certain advantages over the ordinary method.

4. Research Design/Methodology

Let us assume that during a group AHP session m experts are providing PCs of n objects. As a result, we obtain a set of PCM $\{A^{(k)}; k = 1..m\} = \{a_{ij}^{k}: k = 1..m; i, j = 1..n\}$. Based on the classical Cayley's formula, the number of spanning trees we can obtain is $T \le mn^{n-2}$ (equality holds when all PCM are complete). Every basic PC set number q allows us to reconstruct an ideally consistent PCM (ICPCM) and a vector of priorities $\{w_j^q; j = 1..n; q = 1..T\}$. Under the "ordinary" approach ((Lundy et al, 2017), (Bozoki & Tsyganok, 2019)), the formula for aggregate priority calculation is geometric mean (1).

$$w_j^{aggregate} = \left(\prod_{q=1}^{T} w_j^q\right)^{1/T} ; j = 1..n \quad (1)$$

Modified combinatorial method we suggest uses not simple but weighted geometric mean. The weights (or ratings) of basic PC sets reflect completeness, detail, consistency, and compatibility of the respective expert judgments (2).

$$w_j^{aggregate} = \prod_{k,l=1}^{m}\left(\prod_{q^k=1}^{T_k}\left(w_j^{(kq_k l)}\right)^{\frac{R_{kq_k l}}{\sum_{u,p,v} R_{upv}}}\right) ; j = 1..n \quad (2)$$

Multiplicative form of basic PC set (spanning tree) rating is as follows:

$$R_{kql} = c_k c_l s^{kq} s^l / ln\left(\prod_{u,v} max\left(\frac{a_{uv}^{kq}}{a_{uv}^{l}}; \frac{a_{uv}^{l}}{a_{uv}^{kq}}\right) + e - 1\right) \quad (3)$$

k, l are the numbers of experts ($k, l = 1..m$), whose PCM are being compared with each other; c_k, c_l are a-priori values of relative expert competence; k and l can be equal or different; q is the number of ideally consistent PCM copy $q = 1..mT_k$; s^{kq} is the relative average weight of scales in which basic PC set elements are input; it is calculated based on Hartley's formula for signal coding:

$$s^{kq} = \left(\prod_{u=1}^{n-1} log_2 N_u^{(kq)}\right)^{\frac{1}{n-1}} \quad (4); \quad s^k = \left(\prod_{\substack{u,v=1 \\ v>u}}^{n} log_2 N_{uv}^{(k)}\right)^{\frac{2}{n(n-1)}} \quad (5)$$

s^l is the average weight of scales, in which elements of the respective PCM of expert number *l* are input. N is the cardinality of the scale, in which the respective PC is provided.

5. Data/Model Analysis

In the general case the two methods (ordinary and modified) yield different results (coinciding only when the initial individual PCM are ideally consistent and compatible). For instance, in the example from (Kadenko et al, 2019) priorities obtained on the same initial PCMs using ordinary and modified methods amount to (0.590; 0.244; 0.114; 0.052) and (0.564; 0.263; 0.121; 0.052), respectively. The modified method has the listed advantages (consideration of expert data quality, feedback with experts, universality of use). In (Kadenko et al, 2019) we show that simulation of expert estimation process is the best way to empirically compare ordinary and modified methods. Empirical results of simulations indicate that the modified method is more stable to fluctuations of initial expert data than the ordinary one. Simulations are based on the idea of adding some "noise" δ to each element of an initial ICPCM ($a_{ij}^{'} = a_{ij} \pm a_{ij} \cdot \delta/100\%$) and defining the maximum deviation Δ of resulting priorities $\Delta = \max_i(|w_i^{'} - w_i|/w_i) \times 100\%$, where $\{w_i; i = 1..n\}$ is the initial set of weights, based on which the ICPCM is built.

6. Limitations

Results of empirical comparisons of the two versions of the method do not allow us to state that the modified method is more accurate for all test sets of judgments. The limitations are as follows. First, the genetic algorithm we use to calculate the deviations of priorities might, by definition, "skip" significant results. Second, the sensitivity of combinatorial aggregation method depends on several parameters, which are difficult to keep track of: the "noise" level δ and, more important, the diameter of spanning tree graphs, used to reconstruct the ICPCM. For instance, "star-type" spanning trees have a diameter of 2 edges, while "path-type" trees have a diameter of $(n-1)$ edges. So, deviations of ICPCM elements, reconstructed using the respective spanning trees range from 0 to approximately $(n-1)\delta$. For instance, if elements of initial ICPCM A are fluctuated by noise δ, and then some ICPCM A^* is reconstructed based on a spanning tree of diameter $k \in [2, .., (n-1)]$, then the most "deviated" element of A^* is

$$a_{ij}^* = a_{ij}\,(1 \pm \delta)^{k_1}/(1 \mp \delta)^{k_2} \quad (6)$$

where $k_1 + k_2 = k$. These deviations further influence the spanning tree ratings, priorities, and output deviation Δ, making analytical comparison of the two methods a challenge.

7. Conclusions

A modified combinatorial method of PC aggregation in AHP has been suggested. It has several conceptual advantages over the ordinary method, by construction. Empirical results, obtained based on multiple expert session simulations, confirm the advantage of the modified method over the ordinary one in terms of stability and sensitivity. However, analytical comparison of the two versions of the method in terms of sensitivity still presents a problem. Further research will be dedicated to deeper studies of the method's sensitivity based on graph theory.

8. Key References

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