Priority Vector Estimation: Consistency, Compatibility, Precision

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Abstract

Various methods of the priority vector estimation are known in the Analytic Hierarchy Process (AHP). They include the classical eigenproblem method given by Thomas Saaty, developments in least squares and multiplicative approach, robust estimation based on transformation of the pairwise ratios to the shares of preferences, and other approaches. In this paper the priority vectors are completed with validation of data consistency, comparisons of vectors compatibility, and estimation of precision for matrix approximation by vectors. Numerical results for different data sizes and consistency show that the considered methods reveal useful features, are simple and convenient, capable of facilitating practical applications of the AHP in solving various multiple-criteria decision making problems.

Keywords: AHP priority vector estimations, consistency measures, *S*-index and *G*-index of compatibility, precision of fitting.

Introduction

Analytic Hierarchy Process (AHP) is a widely used methodology and a set of methods for solving various problems of prioritization. Founded by Thomas Saaty and developed in numerous works of many authors, it is nowadays one of the main approaches for managers and practitioners who need to apply multiple-criteria decision making for reaching their goals. In this work the term AHP is used not in its whole rich entirety but in a narrower sense as a method of finding local priority vectors by a pairwise comparison matrix. Estimations of priority vectors in the AHP include the classical eigenproblem method (EM) proposed by Saaty (1977, 1980, 1994, 1996, 2005), the least squares (LS) solution and the multiplicative or logarithmic (LN) least squares described in (Saaty and Vargas, 1984, 1994; Lootsma, 1993, 1999), and numerous other modifications (for instance, Lipovetsky, 1996, 2009, 2013; Lipovetsky and Tishler, 1999). Particularly, priority vector robust estimation (RE) not prone to possible inconsistencies in pairwise judgements can be based on the ratio transformation to the shares of preferences and obtained by Markov chain modeling for steady-state probabilities (Lipovetsky and Conklin, 2002, 2015).

The current work presents the results of comparisons between these four methods of EM, LN, LS, and RE using several characteristics of closeness for the obtained solutions, including pair correlations, the so-called Saaty compatibility index (*S*-compatibility) described in (Saaty, 2005; Saaty and Peniwati, 2007), and Garuti compatibility index (*G*-compatibility) described in (Garuti, 2007; Garuti and Salomon, 2011). For different sizes and consistency of the matrices of judgement used in the classical AHP literature, the priority vectors are calculated, their compatibility indices estimated, and characteristics of the matrix fit by the vectors are described. In general, the explored methods are simple and convenient, and can significantly facilitate practical applications of the AHP for optimum solutions in various problems.

The paper is organized as follows: Section II describes the methods of priority estimation, Section III defines the measures of compatibility and quality of fit, Section IV discusses several numerical examples, and Section V concludes on the obtained results.

II. Priority vector estimations

Let us briefly describe several main methods of priority vector estimations. The AHP pairwise priority ratios matrix in general form can be written as follows:

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix}. \tag{1}$$

It is a Saaty matrix of pairwise judgements among n items, elicited from an expert. Each element a_{ij} shows a quotient of preference of the i-th item over j-th item in their comparison, so we have transposed-reciprocal elements $a_{ij} = a_{ji}^{-1}$. A theoretical Saaty matrix of pair comparisons defines each ij-th element as a ratio of the unknown priorities w_i and w_j:

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 \;\ldots & w_1/w_n \\ ----- & -------- & ---- \\ w_n/w_1 & w_n/w_2 \;\ldots & w_n/w_n \end{pmatrix} = w * (\tfrac{1}{w})'. \tag{2}$$

The vector-column w consists of the elements $w_1, w_2, \ldots, w_n$, the vector-row $(1/w)'$ contains the reciprocal values $1/w_1, 1/w_2, \ldots, 1/w_n$, and the right-hand side of the relation (2) shows the outer product of these two vectors (where the prime denotes transposition). From (2), it is easy to find the identical relation $Ww = nw$ for the theoretical matrix and vector. For the obtained matrix (1) a similar relation can be presented as the eigenproblem:

$$A\alpha = \lambda\alpha \;, \tag{3}$$

where the first eigenvector α for the maximum eigenvalue λ defines the vector of priorities. It is the eigenvector method EM of the classical AHP.

Another known way is the least squares estimation for priority vector which can be expressed via the following eigenproblem:

$$(AA')\alpha = \lambda^2\alpha \, . \tag{4}$$

The main vector α yields the priority vector in the LS approach.

The third popular approach to the priority estimation is called multiplicative, or logarithmic technique. It can be reduced to calculating the elements of the priority vector as the geometric means of the elements in each row of the matrix (1):

$$\alpha_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \ . \tag{5}$$

The obtained AHP priority vectors are also standardized by the total of the elements, so a solution is divided by the total of all elements, and the sum of the normalized components equals one:

$$\alpha_{i_{normalized}} = \alpha_i / sum(\alpha_i) \ . \tag{6}$$

It is the priority vector estimation in the LN approach.

For the solution with robust estimation (RE) less prone to possible inconsistencies in the pairwise judgements, let us introduce a theoretical matrix of shares

$$U = \begin{pmatrix} w_1/(w_1+w_1) & w_1/(w_1+w_2) & \dots & w_1/(w_1+w_n) \\ ----- & -------- & & ---- \\ w_n/(w_n+w_1) & w_n/(w_n+w_2) & \dots & w_n/(w_n+w_n) \end{pmatrix}, \tag{7}$$

Each element u_{ij} in (7) is defined as i-th priority in the sum of i-th and j-th priorities:

$$u_{ij} = \frac{w_i}{w_i+w_j} = \frac{w_i/w_j}{1+w_i/w_j} \ . \tag{8}$$

To estimate the priority vector using the matrix (7) we write identical equalities:

$$\begin{cases} \frac{w_1}{w_1+w_1}(w_1+w_1)+\frac{w_1}{w_1+w_2}(w_1+w_2)+\ldots+\frac{w_1}{w_1+w_n}(w_1+w_n)=nw_1 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ \frac{w_n}{w_n+w_1}(w_n+w_1)+\frac{w_n}{w_n+w_2}(w_n+w_2)+\ldots+\frac{w_n}{w_n+w_n}(w_n+w_n)=nw_n \end{cases}. \tag{9}$$

Then with notation (8) we present the system (9) as

$$\begin{cases} (u_{11}+\sum_{j=1}^{n}u_{1j})w_1+u_{12}w_2+\ldots+u_{1n}w_n=nw_1 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ u_{n1}w_1+u_{n2}w_2+\ldots+(u_{nn}+\sum_{j=1}^{n}u_{nj})w_n=nw_n \end{cases}. \tag{10}$$

In the matrix form the system (10) can be written as:

$$(U+diag(Ue))w=nw, \tag{11}$$

where U is the matrix (7), e denotes a uniform vector of n-th order, and $diag(Ue)$ is a diagonal matrix of totals in each row of matrix U.

In the classical AHP, the pair ratios w_i/w_j (2) are estimated by the elicited values a_{ij} (1). Using a_{ij} in (8), we obtain the empirical estimates b_{ij} of the pairs' shares:

$$b_{ij}=\frac{a_{ij}}{1+a_{ij}}. \tag{12}$$

This transformation of the elements of a matrix A (1) yields a pairwise share matrix B with the elements (12). These elements (12) are positive, less than one, and have a property $b_{ij}+b_{ji}=1$. This means that the transposed elements b_{ij} and b_{ji} are skew-symmetrical off the diagonal b_{ii}=0.5, so $b_{ij}-b_{ii}=-(b_{ji}-b_{ii})$.

For empirical Saaty matrix A (1) we have the eigenproblem (3) in place of the theoretical relations (2). Similarly, using the empirical skew-symmetric matrix B (12) in place of theoretical matrix U, we represent the system (11) as the eigenproblem:

$$(B+diag(Be))\alpha=\lambda\alpha, \tag{13}$$

where α as the main eigenvector. It is the RE vector of priority, and its properties have been studied in the works (Lipovetsky and Conklin, 2002, 2015).

III. Measures of Consistency, Compatibility, and Precision

Due to the general methodology of AHP, the so-called consistency index (CI) equals

$$CI = \frac{\lambda - n}{n-1} \tag{14}$$

where λ is the maximum eigenvalue of the matrix in the problem (3), and n is the matrix order. The so-called random consistency index (RI) is a constant tabulated in the AHP for various n, and the consistency ratio (CR) equals the following value:

$$CR = \frac{CI}{RI}\,. \tag{15}$$

A value CR up to 10% is considered as indicating a small inconsistency in the matrix of pairwise comparisons (1), so such a matrix is acceptable, otherwise, with CR>10%, the data could require a reviewing of the elicited judgements.

For comparisons between the obtained solutions, several characteristics can be applied. Among those are the pairwise correlation between the elements of two vectors, which can be reduced to the expression:

$$r(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}} = \frac{\sum_{i=1}^{n}x_i y_i - \frac{1}{n}}{\sqrt{\sum_{i=1}^{n}x_i{}^2-\frac{1}{n}}\sqrt{\sum_{i=1}^{n}y_i{}^2-\frac{1}{n}}}\,, \tag{16}$$

where a bar above the variables denotes the mean values and those equal $1/n$ for the vectors normalized by (6). Without the items $1/n$ for centering (and this value is small for a bigger n) the measure (16) coincides with the cosine as normalized projection of one vector onto another one. The closer is a correlation or cosine to 1, the higher is similarity of two solutions. The cosine values repeat the correlations but are slightly bigger, so for a more conservative measure the correlation can be preferred.

Another good measure of closeness between two vectors is the so-called Saaty compatibility index (*S*-compatibility) described in (Saaty, 2005; Saaty and Peniwati, 2007; Garuti and Salomon, 2011). This index can be found as follows. For two vectors x and y of an n-th order, build a matrix X with its elements defined as quotients $X_{ij}=x_i/x_j$ of the components of the vector x, and a matrix Y with its elements defined as quotients $Y_{ij}=y_i/y_j$ of the components of the vector y. Take the transposed matrix Y' with the elements $Y'_{ij}=y_j/y_i$ and find the Hadamard element-wise product of these two matrices $X*Y'$, then the *S*-index is defined as the normalized total of the elements of this matrix:

$$S = \frac{1}{n^2}\sum_{i,j=1}^{n} X_{ij}Y'_{ij} = \frac{1}{n^2}\sum_{i,j=1}^{n} \frac{x_i}{x_j}\frac{y_j}{y_i}\,. \tag{17}$$

If two vectors coincide, this index equals 1. Within 10% of discrepancy, when $S \leq 1.1$, the vectors are considered as compatible; otherwise, when $S > 1.1$, they are incompatible (Saaty and Peniwati, 2007).

A further development of a compatibility measure in the so-called compatibility index G was proposed in (Garuti, 2007; Garuti and Salomon, 2011) where it was defined as:

$$G = \sum_{i=1}^{n} \frac{min(x_i,y_i)}{max(x_i,y_i)}\frac{x_i+y_i}{2}\,. \tag{18}$$

Due to recommendation in (Garuti, 2007), the values $G<0.9$ correspond to incompatible vectors, otherwise the vectors are compatible.

To check a precision of fit for the pairwise judgements by the priority vector estimate, we can use definition of the elements a_{jk} as quotients of preference between each pair of j-th and k-th items. With a vector-column of priority estimate α, we find its element-reciprocal vector-row $(1/\alpha)'$ and build their outer product by the same pattern as used in (2). With this outer product we find a quality of its fit for the matrix A (1). The standard error (STE) is a measure of the mean distance between the observed and estimated pairwise ratios:

$$STE = \sqrt{\frac{1}{n^2}\sum_{j,k=1}^{n}\left(a_{jk} - \frac{\alpha_j}{\alpha_k}\right)^2} \quad . \tag{19}$$

Another convenient measure of the precision for a matrix approximation by the vectors outer product is the mean absolute error (MAE):

$$MAE = \frac{1}{n^2}\sum_{j,k=1}^{n}\left|a_{jk} - \frac{\alpha_j}{\alpha_k}\right| \quad . \tag{20}$$

The smaller are values of fit (19)-(20), the better is quality of the vector estimate. The measures of STE and MAE can be obtained by using in (19)-(20) only for the off-diagonal pairwise ratios equal or above 1 because they correspond to the elicited quotients of preference, and the reciprocal values below 1 are simply added in completion of the matrix (1) of pairwise judgements.

Besides the characteristics of the residual mean values assessed via standard deviation (19) or absolute deviation (20), the quality of approximation of the pairwise judgements by the obtained priority vectors can be checked by a measure reminding the coefficient of multiple determination R^2 widely used in regression analysis. As shown in (Lipovetsky, 2009) this coefficient can be defined via the observed and estimated paired ratios of the priorities:

$$R^2 = 1 - \frac{RSS}{ESS} = 1 - \frac{\sum_{j,k=1}^{n}\left(a_{jk} - \frac{\alpha_j}{\alpha_k}\right)^2}{\sum_{j,k=1}^{n}\left(a_{jk} - 1\right)^2} \, . \tag{21}$$

In the numerator (21) the residual sum of squares (*RSS*) of the estimated priority deviations from the elicited values is used, and the denominator is presented by the equivalent sum of squares (*ESS*) which assumes all the same preferences $\alpha_j / \alpha_k \equiv 1$. The coefficient (21) shows how much the found priorities outperform the case of absence of preferences among the alternatives. The better is approximation of the paired judgements by the estimated priorities – the closer is *RSS* to zero, so the coefficient of determination R^2 is bigger and closer to one. In absence of preferences $\alpha_j/\alpha_k = 1$, the numerator equals the denominator, and $R^2 = 0$. For the exact fit $a_{jk} = \alpha_j/\alpha_k$ for all judgements, $RSS = 0$, and $R^2 = 1$.

The value R^2 commonly belongs to the interval from 0 to 1, that makes it a very convenient measure for comparison of the priority vectors obtained by different techniques. Only really poor estimates can produce the residual total *RSS* above the value of the equivalent residuals *ESS*, and it would be indicated by the negative R^2. The characteristic (21) corresponds to *STE* measure (19) of squared deviations, but it is possible to build the other estimates, for example, using the MAE residuals (20) as well.

IV. Numerical comparisons for priority estimations

Let us consider numerical examples of the priority estimations for three classical AHP problems.

Example 1: the problem of "Choosing the best home", described in (Saaty and Kearns 1985; Saaty and Vargas, 1994; Saaty, 1996). This matrix is also used for checking some new approaches in (Lipovetsky, 1996; Lipovetsky and Tishler, 1999; Lipovetsky and Conklin, 2002, 2015). The criteria of comparison are: 1 – size of house, 2 – location to bus, 3 – neighborhood, 4 – age of house, 5 - yard space, 6 – modern facilities, 7 – general condition, 8 – financing. The matrix of pairwise comparisons *A* (1) for this problem is presented in Table 1a.

In this example with n=8, the maximum eigenvalue (3) of the matrix in Table 1a equals $\lambda = 9.669$. With the random consistency for this case *RI*=1.41, the consistency index and consistency ratio (14)-(15) are:

$$CI = \frac{9.669-8}{7} = 0.238, \quad CR = \frac{0.238}{1.41} = 0.169. \quad .$$

A value of CR up to 10% is considered as indicating some inconsistency, so the obtained result of 17% can be acceptable with a reservation, when the data could require a reviewing of the elicited judgements, and in (Lipovetsky and Conklin, 2002) it was shown how to identify and to adjust the data in this case.

Table 1a. Example 1: Choosing the best home problem. Pairwise comparison matrix.

item	1	2	3	4	5	6	7	8
1	1	5	3	7	6	6	1/3	1/4
2	1/5	1	1/3	5	3	3	1/5	1/7
3	1/3	3	1	6	3	4	6	1/5
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
6	1/6	1/3	1/4	4	2	1	1/5	1/6
7	3	5	1/6	7	5	5	1	1/2
8	4	7	5	8	6	6	2	1

Several methods of priority estimation for this data are presented in Table 1b. In its upper part, there are four estimates of the priority vector: the classical EM solution (3), the LS estimation (4), the LN technique (5), and the robust estimation RE (13). All the vectors are normalized by the total of their elements equals one (6). Judging by eye, all solutions are very similar by weights, the smallest and the biggest by importance are the age and financing of the house, the items 4 and 8, respectively.

For comparison between the obtained four priority vectors we apply the measures (16)-(18) of Correlations, S-compatibility, and G- compatibility, presented after the vectors in three matrices in Table 1b. Judging by correlations, all the vectors are close enough by their structure, and LS is a bit further from the three others. The measure of *S*-compatibility proves that the EM, LN, and RE vectors are similar, within less than the required threshold of 10% of *S*-index deviation from one. The more sensitive *G*-compatibility demonstrates that the pair of EM and LN vectors are close with G=0.927, and two vectors LN and RE are close with G=0.912, which are the values above the threshold 0.9 needed for viewing the corresponding vectors as compatible.

Table 1b. Example 1: Choosing the best home problem. Priority vector estimations.

item	EM	LS	LN	RE
1. size of house	0.173	0.199	0.175	0.150
2. location to bus	0.054	0.100	0.063	0.054
3. neighborhood	0.188	0.148	0.149	0.141
4. age of house	0.018	0.017	0.019	0.022
5. yard space	0.031	0.045	0.036	0.037
6. modern facilities	0.036	0.065	0.042	0.041
7. general condition	0.167	0.184	0.167	0.163
8. financing	0.333	0.242	0.350	0.392
Correlations				
EM	1	0.935	0.988	0.972
LS	0.935	1	0.933	0.881
LN	0.988	0.933	1	0.991
RE	0.972	0.881	0.991	1
S-compatibility				
EM	1	1.113	1.015	1.028
LS	1.113	1	1.071	1.122
LN	1.015	1.071	1	1.010
RE	1.028	1.122	1.010	1
G-compatibility				
EM	1	0.774	0.927	0.865
LS	0.774	1	0.809	0.742
LN	0.927	0.809	1	0.912
RE	0.865	0.742	0.912	1
Precision				
STE	2.071	1.849	1.813	1.831
MAE	1.079	1.083	0.958	0.934
R^2	0.423	0.540	0.558	0.549

The last segment at the bottom of Table 1b displays the precision by (19)-(21) for each vector solution. By the minimum standard error STE – the best model is LN, and by the mean absolute error MAE – the best model is RE. The values of MAE suggest also that an average deviation from the observed pair judgements evaluated by the obtained quotients from a priority vector is not more than one unit. The coefficient of multiple determination R^2 in the last row of Table 1b shows by its maximum values that the models LN and RE outperform the other two models, though all R^2 values are not high, which indicates a difficulty in approximation of inconsistent judgements by a priority vector in any estimation.

Example 2: the problem of "Distance from Philadelphia" is one of the first AHP problems described by Saaty (1977). The remoteness of six cities from Philadelphia was estimated by the criterion: for each pair of cities, how many times farther the more distant city is located from Philadelphia than the nearer one is? The elicited data is presented in Table 2a.

Table 2a. Example 2: Distance from Philadelphia problem. Pairwise comparison matrix.

Airport	CAI	TYO	ORD	SFO	LGW	YMX
Cairo.CAI	1	0.333	8	3	3	7
Tokyo.TYO	3	1	9	3	3	9
Chicago.ORD	0.125	0.111	1	0.167	0.2	2
SanFrancisco.SFO	0.333	0.333	6	1	0.333	6
London.LGW	0.333	0.333	5	3	1	6
Montreal.YMX	0.143	0.111	0.5	0.167	0.167	1

The maximum eigenvalue (3) in this example equals $\lambda = 6.454$. The random consistency for n=6 is RI=1.24, then consistency index and consistency ratio (14)-(15) are:

$$CI = \frac{6.454-6}{5} = 0.091, \quad CR = \frac{0.091}{1.24} = 0.073 \ .$$

The value CR=7.3% lesser than 10% permits to conclude that the data on pair judgements is sufficiently consistent.

Table 2b presents the results of priority estimations for this example, and it is organized as the previous Table 1b, but with one additional column of the actual shares of distances known in this case.

Table 2b. Example 2: Distance from Philadelphia problem. Priority vector estimations.

City	EM	LS	LN	RE	actual
1. Cairo	0.262	0.254	0.260	0.239	0.278
2. Tokyo	0.397	0.305	0.399	0.447	0.361
3. Chicago	0.033	0.047	0.035	0.034	0.032
4. San Francisco	0.116	0.186	0.116	0.104	0.132
5. London	0.164	0.184	0.163	0.147	0.177
6. Montreal	0.027	0.024	0.027	0.029	0.019
Correlations					
EM	1	0.943	1.000	0.990	0.991
LS	0.943	1	0.941	0.898	0.973
LN	1.000	0.941	1	0.991	0.990
RE	0.990	0.898	0.991	1	0.962
actual	0.991	0.973	0.990	0.962	1
S-compatibility					
EM	1	1.064	1.000	1.009	1.024
LS	1.064	1	1.064	1.106	1.045
LN	1.000	1.064	1	1.008	1.027
RE	1.009	1.106	1.008	1	1.060
actual	1.024	1.045	1.027	1.060	1
G-compatibility					
EM	1	0.821	0.993	0.900	0.914
LS	0.821	1	0.820	0.753	0.854
LN	0.993	0.820	1	0.905	0.908
RE	0.900	0.753	0.905	1	0.823
actual	0.914	0.854	0.908	0.823	1
Precision					
STE	1.390	1.340	1.333	1.523	2.295
MAE	0.696	0.862	0.686	0.790	1.012
R^2	0.794	0.809	0.810	0.753	0.438

We see that in general the vectors are similar and each one makes sense as proportionally scaled distances from Philadelphia to other cities in the USA, as well as to other countries and continents. The pair correlations also show that the vectors are closely related to the actual distances, and the same is supported by *S*-compatibility index. *G*-compatibility indicates that EM and LN vectors are compatible with the actual shares of distances. The precision of the reproduction of the judgement matrix is high, especially by the LS and LN methods. Precision measured by *STE*, *MAE*, and R^2 of the actual distances is the worst one within the other values in the last rows of Table 2b. It means that the pair judgements on distances correspond rather to the priority vectors than to the actual distance shares. Therefore, in this data we do not need to use the actual data in considering compatibility among the vectors.

Example 3. The data for this problem is given in (Whitaker, 2007) where the area of five geometric figures were compared – see the matrix of pair judgements in Table 3a.

Table 3a. Example 3: Geometric figures' area problem. Pairwise comparison matrix.

figure	Circle	Triangle	Square	Diamond	Rectangle
Circle	1	9	2.5	3	6
Triangle	0.111	1	0.2	0.286	0.667
Square	0.4	5	1	1.7	3
Diamond	0.333	3.5	0.588	1	1.5
Rectangle	0.167	1.5	0.333	0.667	1

The maximum eigenvalue of this matrix is $\lambda = 5.026$. The random consistency for *n*=5 is *RI*=1.12, so consistency index and consistency ratio (14)-(15) equal the following values:

$$CI = \frac{5.026-5}{4} = 0.006, \quad CR = \frac{0.006}{1.12} = 0.006\ .$$

The *CR*=0.6% proves a very high level of consistency of this data. It can be explained by the used pairwise ratios where not only the integer numbers but also the rational numbers (like 2.5 or 3.5) were permitted in the preference evaluation.

Table 3b presents the priority estimation results for this example, and it is organized as Table 2b, with the additional column of the actual shares of the areas measured for these figures.

Table 3b. Example 3: Geometric figures' area problem. Priority vector estimations.

	EM	LS	LN	RE	actual
1. Circle	0.488	0.464	0.487	0.496	0.470
2. Triangle	0.049	0.050	0.049	0.049	0.050
3. Square	0.233	0.248	0.233	0.225	0.240
4. Diamond	0.148	0.159	0.148	0.147	0.140
5. Rectangle	0.082	0.078	0.082	0.083	0.090
Correlations					
EM	1	0.998	0.999	0.999	0.999
LS	0.998	1	0.998	0.995	0.998
LN	0.999	0.998	1	0.999	0.999
RE	0.999	0.995	0.999	1	0.998
actual	0.999	0.998	0.999	0.998	1
S-compatibility					
EM	1	1.003	1.000	1.000	1.003
LS	1.003	1	1.003	1.004	1.007
LN	1.000	1.003	1	1.000	1.003
RE	1.000	1.004	1.000	1	1.003
actual	1.003	1.007	1.003	1.003	1
G-compatibility					
EM	1	0.948	0.999	0.982	0.955
LS	0.948	1	0.948	0.931	0.953
LN	0.999	0.948	1	0.982	0.956
RE	0.982	0.931	0.982	1	0.940
actual	0.955	0.953	0.956	0.940	1
Precision					
STE	0.253	0.195	0.253	0.289	0.279
MAE	0.145	0.110	0.145	0.162	0.165
R^2	0.987	0.992	0.987	0.983	0.985

All the vector estimates in this data look practically coinciding, the pair correlations are very high, and both *S*- and *G*- indices prove compatibility among the estimates and with the actual observations. The precision measured by *STE*, *MAE*, and R^2 characteristics demonstrates a high quality of the data fit by any of the estimated vectors of priority and by the actual values as well.

V. Conclusions

The paper considered several methods of priority vector evaluation in the AHP. They include the classical eigenproblem method, least squares, multiplicative or logarithmic approach, and a robust estimation based on transformation of the pairwise ratios to the shares of preferences. Together with estimation of the vectors, validation of data consistency and comparison of vectors by correlations, *S*- and *G*- compatibility indices were completed too. Numerical results for different data sizes and consistency indices demonstrate that all the methods produce compatible results for the consistent data, otherwise a discrepancy between different methods of the priority estimation could be observed. Therefore, the data consistency should be always proved before the vector evaluation.

Another important conclusion concerns the precision assessment for the data matrix approximation by the obtained priority vectors. Any regular statistical modeling requires a verification of the produced results by some quality characteristics. For example, in regression analysis, such measures as the residual standard error *STE*, mean absolute error *MAE*, and coefficient of multiple determination R^2 are commonly employed. Applying them in the AHP environment can enrich the evaluation and interpretation of the results on priority modeling, and it is demonstrated on the numerical estimations performed in the paper. For instance, in the data of Example 1 with a low consistency, the R^2 values are also not high which indicates a difficulty of approximation of inconsistent judgements by a priority vector in any estimation, and by *MAE* values a mean deviation of the quotients of a priority vector's elements from the observed pair judgements could be as big as one unit. In Example 2 with a good consistency, the precision of the reproduction of the judgement matrix by the found priority vectors is high enough, although at the same time the actual distances occurred to yield the worst vector for approximation of the elicited pairwise priority matrix Thus, in this data we should not use the actual data on distances in checking its compatibility with the obtained estimates of the vectors. Example 3 with a perfect consistency yields all the vectors of high compatibility, and of a great quality of the elicited judgements reconstruction by each vector's quotients of preference.

Resuming, the considered methods of priority vector estimation and characteristics of their quality are convenient and helpful in practical applications of the AHP for solving various multiple-criteria decision making problems.

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