THEORY AND APPLICATION OF THE ANALYTIC HIERARCHY PROCESS

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SESSION ABSTRACT

This session discusses the most fundamental features of the Analytic Hierarchy Process, i.e., pairwise comparison and its aggregation process, and the most practical application, e.g., diagnosis for human preference. In the AHP procedure, pairwise comparison and its aggregation process play key roles in quantifying human perception. The eigenvector method and the use of linear scale, as well as the l_1-normalization of the outputs are considered as the standard for the process. While, not a few criticisms exist, which propose other non-linear scales, or different approaches to aggregation. Paper proposals 2 and 3 deal with these issues. On the other hand, the AHP has come into wide use to various fields, in conjunction with the development of software, e.g., expertchoice®, because of its powerful and flexible decision making process, and its versatility and compatibility. Among the enormous amount of applications, quantification of human perception and its application to diagnosis procedure for user preference in supply chain management is one of the most prosperous fields. Paper proposal 1 introduces a case.

A HYBRID DIAGNOSIS PROCEDURE FOR OPTIMIZING THE SPECIFICATION OF BTO PRODUCTS

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ABSTRACT

The objectives of this paper are to propose a diagnosis process of user's preference for a build-to-order (BTO) product. Manufacturing companies need to investment strategically for their development due to increasing focus on corporate social responsibilities, compliance and sustainability. However, determining the specification of a BTO product is a complicated task, because of subjective factors entering into the evaluation of necessary and sufficient specification of the system. Consequently, the choice of appropriate specification often lacks transparency and traceability in the process. This paper addressed this issue by combining cost-benefit analysis and the AHP. Wastewater treatment system for a chemical company is considered as one of BTO products, and a case study in the company was carried out to demonstrate the applicability of the procedure. The results of this paper show some evidence that the diagnosis process proposed in this paper succeeded in quantifying user's preference for potential systems.

Keywords: build-to-order products, cost-benefit analysis, Analytic Hierarchy Process

1. Introduction

Decision making for strategic investment in build-to-order (BTO) is complicated, particularly in manufacturing companies. In optimizing specification of the products, managers must evaluate and choose products architectures, which are usually costly and surrounded by uncertainty over the likely effects of the different architectures available. The difficulties arise mainly from intangible factors, such as the manager's judgment of criteria that enter into the evaluation and choice of appropriate architecture, given the rapidly changing technological environment. Furthermore, companies have their "preference" for the products, which are often expressed as subjective information. Thus, decision making regarding strategic investment, relying heavily on experience, knowledge, as well as intuition, often lacks transparency and traceability.

2. Literature Review

Significant amounts of literature in technology investment and selection exist. Freitas et al. (2000) applied an expert system to develop a conceptual design of industrial wastewater treatment process, but the process in the application was a black box; the selection process thus lacked transparency. Some papers adopted optimization methods to solve the problem of wastewater treatment system design, which assumed that all information on the system design would be given quantitatively for designers in solving the problem (Noble & Tanchoco, 1995). In designing complex BTO products, however, intangible factors in the design process should be taken into account. Stehna and Bergströmb (2002) proposed customer-oriented design process of products which could be applied to the design of wastewater treatment system; the approach, however, did not explicitly take user's subjective preference into design.

Therefore, decision makers are unaware of what factors have been considered and what trade-offs have been taken; it is difficult to convince decision makers to trust the expert system's solution. Safety supervisors in manufacturing companies thus face huge challenges in designing the treatment system and selecting the appropriate technology suppliers. On the other hand, selecting and designing a sustainable treatment system is vital because this is not a one-off investment; the financial sustainability is also a very important factor. A trade-off between purchasing cost and operational cost needs to be resolved in relation to the expected life of the plant (Jianga, et al., 2002). Thus, it is substantial for managers to optimize the specifications of the products so as to satisfy the preference for the products.

3. Objectives

This paper proposes a hybrid diagnosis procedure by combining the cost-benefit-analysis (B/C analysis) and the Analytic Hierarchy Process (AHP) for optimizing the specification of build-to-order (BTO) products of a manufacturing company. In the approach, subjective factors in a decision making process are quantified using the AHP. The approach combining B/C analysis and the AHP consists of a sequence of transparent steps to provide clarity of thought into the evaluation and selection process for a company. The result evaluating not only the treatment systems but also the decision criteria, therefore, fully justifies the final decision. An additional benefit of the justification is that the rationale behind each decision is captured and can then be used as the basis of an overall justification.

4. Research Design

This section describes the diagnosis procedure optimizing the specification of BTO products, in which a wastewater treatment system (WTS) of a chemical company is an example of BTO products. One of the most difficult features in determining the specification of a WTS is how to direct design efforts. Since the making decisions with large amount of investment were made based on the consensus of executives of the company, the decision process would be sometimes inconsistent due to subjective factors among the executive' preference for the system.

WTS purifies wastewater before discharging it into environment so as to satisfy the allowable limits defined by the law. In order to purify wastewater, the system equips several subsystems, each of which has different functions and performances. By combining these subsystems, the system satisfies the allowable limits; while, the profile of the combination is not unique. On the other hand, the regulation set by the legislation has a broad range of items concerning the wastewater treatment, which define the specifications of WTS. Each item can be completely specified as objective information, while, as a user of WTS, the company has its own preference for the system, which includes subjective information. This information relates to each aforementioned specification and is represented as intangible information. Diagnosis procedure of user's preference for a new BTO WTS is thus substantial in optimizing the specification of the system for the company.

The diagnosis procedure needs to evaluate all potential alternatives in light of cost and benefit, which integrates objective data and subjective preference for the specification of WTS. Thus, the process consists of two following main phases: evaluating potential alternatives in light of cost and benefit; collecting information on the preference for the specification of WTS. Since the performance of abovementioned subsystem is clarified, a set of criteria, *c*, evaluating the system are defined as follows in this paper: "*A*" (Area of installation); "*I*" (Initial cost); "*R*" (Running and maintenance costs); "*S*" (Leakage amount of Suspended Soil); "*B*" (Leakage amount of Biological Oxygen Demand); "*O*" (Leakage amount of H_2S). Those criteria, and subsequent indicators defined below were determined based on the discussion with designers of a supplier of WTS.

The following indicators are employed in the B/C analysis. Benefits are defined by the reciprocal values of $A_{*,i}$, $S_{*,i}$, $B_{*,i}$, and $O_{*,i}$, and costs are defined by the actual values of $I_{*,i}$ and $R_{*,i}$, where * (*=I: high, II: intermediate, III: low) and *i* (*i*=0, …, 8) respectively denote load per area of Rotating Contactor (RC) in the system, and the number of RC of the treatment system, each of which is indexed using the value when *=I and *i*=0 as a benchmark (set as 1). On the other hand, users of the WTS have their own preference for the system on the premise that the alternatives of the system satisfy the regulation. The user, therefore, would make decisions on which system architecture to order from among the potential alternatives. However, user's preference is often expressed as subjective information and users are sometimes caught in a dilemma of conflicting preference. In this paper, user's preference is supposed to be represented upon *A*, *I*, *R*, *S*, *B*, and *O*. Collecting information on user's preference is carried out by using the AHP, so as to transform such intangible information into quantitative form. B/C function is then formulated for analyses, in which w_c denotes user's preference for the criterion *c*. Therefore, the T-score of the criterion *c* (*c*=*A*, *I*, *R*, *S*, *B*, *O*) can be represented by the following and denoted as $c^w_{*,i}$, where $\mu(c_{*,i})$ and $\sigma(c_{*,i})$ are respectively denote the average and the standard deviation of $c_{*,i}$.

$$c^w_{*,i} := 50 + 10\{c_{*,i} - \mu(c_{*,i})\} w_c / \sigma(c_{*,i}). \quad (1)$$

Based on (1), the "weighted" B/C function can be defined by the following formula which calculates each alternative's weighted performance reflecting user's preference.

$$WCB_{*,i} := \{1/A^w_{*,i} + 1/S^w_{*,i} + 1/B^w_{*,i} + 1/O^w_{*,i}\} / \{I^w_{*,i} + R^w_{*,i}\}. \quad (2)$$

5. Model Analysis

A wastewater treatment system company X (Co.X) is a supplier of a wastewater treatment system in Japan. Co.X develops various types of the treatment system with RC,

meeting the demands from great many manufacturing companies. The company's name, X, cannot be disclosed due to confidentiality obligation; on the other hand, all information stated in this case study is the real in the company.
There are three major indicators for the wastewater treatment: (*i*) total amount of leaked BOD, (*ii*) concentration of leaked SS, and (*iii*) concentration of generated H_2S. Designers in Co. X need to design the treatment system to purify the wastewater to meet the required legislation level with three core subsystems in WTS. The decision process is complicated as it requires selecting the most appropriate combination of subsystems and deciding the architecture of the treatment system with satisfying user's preference. The criteria assess the potential benefits of the new system, its alignment with the user's strategy, its impact on identified objectives, and its failure risks. With this approach, a number of alternatives of the treatment system are designed and evaluated by the set of criteria.

By using the weighted B/C analysis, we can develop a diagnosis procedure for customers of Co.X. Table 1 shows the results of the analyses, which summarizes rankings of the same alternatives with different users' preference. In the analyses, by changing the values of pairwise comparisons so as to emphasize the degree of the importance of a criterion, abovementioned priorities are artificially generated. As shown in the table, the rankings of alternatives differ drastically based on the user's preference, which leads to the different choice of the architecture of the treatment system.

Table 1
Results of sensitivity analyses

Prioritized criteria		$1/A_{*,i}$	$I_{*,i}$,	$R_{*,i}$,	$1/S_{*,i}$	$1/B_{*,i}$	$1/O_{*,i}$
*	*i*						
I (High)	0	**1**	**2**	13	17	8	**1**
	1	9	12	20	*22*	10	4
	2	7	9	9	13	5	5
	3	14	15	14	11	4	7
	4	18	16	*22*	21	9	14
	5	20	18	18	12	6	15
	6	*22*	21	21	9	**2**	20
	7	*23*	*23*	*23*	10	7	*23*
	8	*24*	*24*	*24*	7	**3**	*24*
II (Intermediate)	0	**2**	4	15	19	15	**2**
	1	11	13	19	*24*	21	18
	2	6	7	17	*23*	19	16
	3	12	10	11	14	16	13
	4	15	14	16	16	18	*22*
	5	17	17	10	8	14	21
	6	19	19	6	5	13	17
	7	21	*22*	5	4	11	19
	8	16	20	**2**	**1**	**1**	8
III (Low)	0	**3**	5	12	18	*22*	**3**
	1	4	**3**	7	15	*23*	11
	2	5	**1**	8	20	*24*	12
	3	8	6	4	6	20	10
	4	13	11	**3**	**3**	17	9
	5	10	8	**1**	**2**	12	6

Traditional approaches could not address the WTS selection problem with taking subjective factors into consideration. The approach combining B/C analysis and the AHP proposed in this paper provides a framework for considering the impact of each trade-off decision on the criteria, and develops a justification path for the management of companies. Co.X management is satisfied with the diagnosis procedure that can address different users' preference. By using this procedure, the company would be able to address various users' preference and to optimize the architecture of the treatment system. The results shown in Table 1 are significant for Co.X, which helps its promotion of the treatment system to potential customers.

6. Limitations

In this paper, the indices of the specifications of products are defined not so plausible way. How to define the indices of the specifications of the system, such as the reciprocal values for the benefits of Flexibility, is an open-ended question. Further research is needed to explore design improvement and technology selection in more complex industrial wastewater treatment systems.

7. Conclusions

This paper proposes a diagnosis procedure of user's preference for BTO products for a manufacturing company, which integrates both objective and subjective information on the system. The approach not only satisfies the required legislation level, but also reflects user's preference to the design of the system, which makes decision path more transparent than before. The case study demonstrates the applicability of the approach that supports designing BTO products. A B/C analysis is a systematic approach to evaluate the system performance, while the application of the AHP is a simple approach to transform the subjective information into objective information. The proposed approach, therefore, enables safety supervisors to deal with this information on the same horizon. By providing clarity to the analysis process, the decision making results in transparent and traceable.

8. Key References

Freitas, I.F.S., Costa, C.A.V. & Boaventura, R.A.R. (2000). Conceptual design of industrial wastewater treatment processes: primary treatment. *Computers and chemical Engineering, 24,* 1725-1730.

Jianga, R., Zhangb, W.J. & Jia, P. (2004). Selecting the best alternative based on life-cycle cost distributions of alternatives. *International Journal of Production Economics, 89(1),* 69-75.

Noble, J.S. & Tanchoco, J.M.A. (1995). Marginal Analysis Guided Design Justification: A Material Handling Example. *International Journal of Production Research*, 33(12), 3439-3454.

Stehna, L. & Bergströmb, M. (2002). Integrated design and production of multi-storey timber frame houses – production effects caused by customer-oriented design. *International Journal of Production Economics, 77(3),* 259-269.

RELATIONSHIP BETWEEN THE ANALYTIC HIERARCHY PROCESS AND WEIGHTED SUMMATION

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ABSTRACT

Weighted summation, which is also called weighted average method, is a famous and simple multi-criteria evaluation method. This method is essentially applied for items with absolute evaluation values like marks or dollars. On the other hand, there are various cases where we want to evaluate items without such values. The AHP is helpful in these cases. The purpose of this paper is to clarify the relationship between the AHP and weighted summation by introducing weighted summation ratio method.

The AHP is similar to weighted summation, but these two methods are different from each other. Indeed, we need some precautions when evaluating items by absolute numbers with respect to criteria. This was already pointed out by T.L. Saaty, the founder of the AHP/ANP.

In this paper, first I extend weighted summation to weighted summation ratio method in order to apply it to items without absolute evaluation values. I also show the validity of it with fundamental mathematics. Weighted summation ratio method is a framework and needs certain methods to guess relative evaluation values of items with respect to criteria and adjust them. Therefore, after that, I propose the method with paired comparisons in the AHP/ANP to the former and logarithmic least square method to the latter. I show a numerical example to explain this method. The relationship between the AHP and weighted summation is shown in the process.

By the way, we might be able to use a supermatrix in the ANP as weighted summation ratio method. However, these two methods are different from each other since it doesn’t take impacts or influences between clusters or items in cluster into consideration like weighted summation. I don’t deal with the relationship between the ANP and weighted summation ratio method.

Keywords: weighted summation, weighted average, relative evaluation.

1. Introduction

Weighted summation, which is also called weighted average method, is a famous and simple multi-criteria evaluation method. This method is essentially applied for items with absolute evaluation values like marks or dollars. On the other hand, there are various cases where we want to evaluate items without such values. The Analytic Hierarchy Process (AHP) is helpful in these cases.

The AHP is similar to weighted summation, but these two methods are different from each other. Indeed, we need some precautions when evaluating items by absolute numbers with respect to criteria. This was already pointed out by T.L. Saaty, the founder of the AHP/ANP. This is one of factors by which AHP users are confused.

The purpose of this paper is to clarify the relationship between the AHP and weighted summation by introducing weighted summation ratio method. Then I show the validity of it with a fundamental theory in set theory in mathematics. This method is a framework and needs extra methods itself. So after that I propose concrete methods including such extra methods to actually use it. Finally, I show a numerical example to explain this method. The relationship between the AHP and weighted summation is shown in the process.

By the way, we might be able to use the Analytic Network Process (ANP) as weighted summation ratio method. However, weighted summation ratio method doesn't take impacts or influences between clusters or items in cluster into consideration like weighted summation. So I don't deal with the ANP in this paper.

2. Literature Review

The AHP is similar to weighted summation, although these two methods are essentially different from each other. The AHP is often introduced and explained like weighted summation. Indeed, the AHP has a mode corresponding to weighted summation, the Absolute mode (Saaty, 2006). These approaches are helpful to understand the concept of the AHP for the beginners, but seem to induce misunderstanding at the same time.

For this problem, Schoner, Wedley and Choo proposed the concept of linking pins between items over all of criteria (Schoner, Wedley & Choo, 1993). This adjusts weights of them in accordance with how to compare items with respect to criteria. Furthermore, this concept can systematically treat various techniques to connect relative evaluation values of items under all criteria proposed by researchers (Wedley, 2009).

In this paper, I approach to this problem from the point of view of weighted summation. Weighted summation ratio method introduced in this paper is similar to linking pin. However, these are different from each other since it calculates relative evaluation values table of items with respect to criteria directly using extra questions.

Saaty, T.L. (2006). *Fundamentals of decision making and priority theory, 2nd ed*. Pittsburgh, PA: RWS Publications.

Schoner, B., Wedley, W.C. & Choo, E.U. (1993). A Unified Approach to AHP with Linking Pins. *European Journal of Operations Research, Vol.64,* 384-392.

Wedley, W.C. (2009). Issues in AHP/ANP: Linking and aggregating relative ratio scales. *Proceedings of JSAHP2009*, 17-37.

3. Hypotheses/Objectives

The purpose of this paper is to clarify the relationship between the AHP and weighted summation. For this, first I extend weighted summation to weighted summation ratio method, which can deal with items with relative evaluation values with respect to criteria. This is a framework and needs certain methods to guess relative evaluation values of items with respect to criteria and adjust them. So after that I propose weighted summation ratio method with paired comparisons in the AHP/ANP and logarithmic least square method in order to actually use it. The relationship between the AHP and weighted summation is shown in the process.

4. Research Design/Methodology

I introduce weighted summation ratio method in order to clarify the relationship between the AHP and weighted summation. To extend weighted summation to weighted summation ratio method, I use an elementary theory of set theory in mathematics.

Indeed, weighted summation ratio method deals with relative values, so we face to the same problem as fractions as follows. Addition of two fractions b/a and d/c is defined by (bc+ad)/(ac). For example, we have 2/5+3/7=(2*7+5*3)/(5*7)=29/35. Now we take 4/10 and 9/21 instead 2/5 and 3/7, respectively. Then we have 4/10+9/21=(4*21+10*9)/(10*21)=174/210. On the other hand, we know that 174/210=29/35. Consequently, we don't have to consider which fraction is chosen out of the set of fractions with the same values as a representative. Such definition is called "well-defined". See (Lang, 1995) for details. This is a validity of choosing an irreducible fraction as a representative in fractions.

This "well-defined" property is guaranteeing that calculation is consistent. If the definition of the calculation isn't well-defined, we can't do well. In this paper, I show that weighted summation ratio method is based on well-defined property of relative evaluation values tables. Saaty pointed out the characteristics of fractions in (Saaty, 2006), but didn't make reference about the well-defined property for evaluation tables.

After introducing weighted summation ratio method, I propose weighted summation ratio method with paired comparisons in the AHP/ANP (Saaty, 1996) and logarithmic least square method. Then I show why we use logarithmic least square method from the point of view of an optimization problem. The difference between the AHP and weighted summation becomes clear by considering a concrete procedure of this weighted summation ratio method.

5. Data/Model Analysis

Examples which show the difference between the AHP and weighted summation have already been well-known. I show a numerical example in order to explain weighted summation ratio method with paired comparisons in the AHP/ANP and logarithmic least square method.

This example is selection of the goods for the female fans of Japanese professional baseball by a female fan. This problem is described with three-level-hierarchy as follows (Figure 1):

Problem: Select the most favorite goods.
Criteria: Price, Cuteness, Everyday use and Color.
Items: Baseball uniform, Baseball cap, T-shirt, Accessory and Bag.

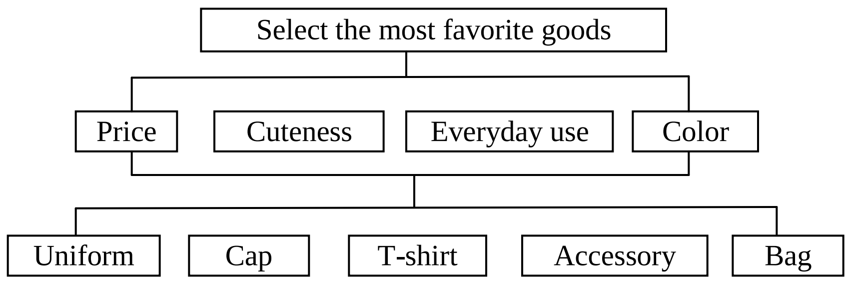

Figure 1 Hierarchy

It is expected that these items are evaluated like weighted summation, but they don't have absolute numbers with respect to criteria. On the other hand, as is well-known, paired comparisons in the AHP/ANP is perfect to this problem. So I use weighted summation ratio method with paired comparisons in the AHP/ANP and logarithmic least square method. This problem was calculated by a female student, who likes Japanese professional baseball.

6. Limitations

Weighted summation ratio method needs extra methods to actually use. This means that this method is just a framework. Moreover, we don't know that weighted summation ratio method with paired comparisons in the AHP/ANP and logarithmic least square method is the best out of all weighted summation ratio method.

For example, we might be able to use a supermatrix in the ANP as weighted summation ratio method. However, I note that the purpose of weighted summation ratio method is different from that of the ANP. In fact, this method doesn't take any influence between clusters or items within a cluster into consideration at all like weighted summation.

I don't fully recognize the merit of weighted summation ratio method in everyday life. For example, if we can get values like absolute numbers for all items, it might be sufficient to use weighted summation. In this paper, I introduced this method in order to clarify the relationship between the AHP and weighted summation. So this method might be useful only when combining to the AHP/ANP.

7. Conclusions

In this paper, I clarify the relationship between the AHP and weighted summation by introducing weighted summation ratio method. Furthermore, I proposed weighted summation ratio method using paired comparisons in the AHP/ANP and logarithmic least square method for actual problems.

Finally, we might be able to apply a supermatrix in the ANP to weighted summation ratio method. In this paper I supposed independence between items or criteria like weighted summation. Therefore, I didn’t deal with the ANP. I leave the relationship between a supermatrix or the ANP and weighted summation ratio method as problems which should be solved in the near future.

8. Key References

Lang, S. (1995). *Algebra, 3rd ed*. U.S.A.: Addison Wesley.

Saaty, T.L. (1996). *The Analytic Network Process*. Pittsburgh, PA: RWS Publications.

Saaty, T.L. (2006). *Fundamentals of decision making and priority theory with the Analytic Hierarchy Process, 2nd ed*. Pittsburgh, PA: RWS Publications.

AN ALGEBRAIC REPRESENTATION FOR COMPARISON METHODS OF AHP

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ABSTRACT

In this paper, we propose a simple algebraic representation for comparison methods of AHP. The representation is associative relation between importances of elements and consists of basic arithmetic operations.

First, we define a ratio, which is estimated by decision makers with comparing importances of elements, as a partial differentiation of importances. And we construct systems of differential equations. Algebraic representations of the importances are derived as formal solutions of the equations.

We analyze the pairwise comparison methods with the representations in section 3. A validity of using eigenvectors in the method is illustrated by a particular solution of the equations.

In section 4, we describe the ternary comparison method, which is a variety of visual analog scaling, with modifying the definition of partial differentiations. We also represent importances as a system of differential equations and derive algebraic representations of them by solving the system. We, probably, first introduce clearly into AHP's field and analyze the method.

Finally, we discuss the applications of the representations.

Keywords: pairwise comparison method, ternary comparison method, ternary diagram

1. Introduction

Pairwise comparison method is a primitive procedure in AHP [Saaty, 1980]. Decision makers construct importances of elements from ratios between them using the method. Let $a_1, \cdots, a_n$ be elements, and x_i be an importance of an element a_i. Decision makers want to obtain x_i, but they can only estimate ratios x_i/x_j. The pairwise comparison method is a procedure whose input is a set of the ratios and whose output is a set of importances of elements. In the procedure, at first, decision makers estimate ratios for all pairs of elements. Let r_{ij} be an estimated value of a ratio x_i/x_j. These values are arranged into a pairwise comparison matrix R, which is $n \times n$ square matrix and its element in i-th row j-th column is r_{ij}. Saaty said that the importances which decision makers want is obtained as the principal eigenvector of R; an element $\widehat{x_i}$ of a vector $\widehat{\boldsymbol{x}} = [\widehat{x_1}, \cdots, \widehat{x_n}]^t$ which holds $\mathrm{R}\widehat{\boldsymbol{x}} = \lambda_{max}\widehat{\boldsymbol{x}}$ is an approximation of $\widehat{x_i}$. Harker and Vargas[Harker and Vargas, 1987] discussed reasons why we can regard the vector as the approximation of importances. Their illustrations, however, are correct but quite difficult because of analyses of eigenvectors. In decision making process, we have to make decision makers understand intuitively usefulness of the methods.

In this paper, we propose a representation which simply illustrates validity of calculations for the pairwise comparison method, and we extend the representation for other comparison method, which is ternary comparison.

2. Hypotheses

We presume that an importance x_i of an element a_i can be represented in a multi-valued function whose arguments are a_j, $j \neq i$;

$$\mathrm{x}_i \equiv x_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \quad i = 1, \cdots, n. \tag{1}$$

A ratio r_{ij} of importances x_i and x_j is estimated by decision makers. We put more assumption that the ratio is a partial differentiation of these functions;

$$r_{ij} \equiv (n-1)\frac{\partial x_i}{\partial x_j}. \tag{2}$$

There is a term $(n-1)$ in the equation (1), because decision makers estimate the ratio of x_i as a single-valued function whose argument is x_i in spite of former assumption that the function is $(n-1)$-valued function.

3. An analysis of the pairwise comparison method

We can write the pairwise comparison matrix R as follows:

$$\mathrm{R} = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} = (n-1)\begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} = (n-1)\partial\boldsymbol{x}\underline{\partial\boldsymbol{x}}^t \tag{3}$$

where $\partial\boldsymbol{x} = [\partial x_1, \cdots, \partial x_n]^t$, and $\underline{\partial\boldsymbol{x}} = [1/\partial x_1, \cdots, 1/\partial x_n]^t$.

Let $d\boldsymbol{x} = [dx_1, \cdots, dx_n]^t$, and let us consider a product $\mathrm{R}d\boldsymbol{x}$. Combining the formula of total differentiation, we obtain a relation:

$$d\boldsymbol{x} = \left(\partial \boldsymbol{x}\, \underline{\partial \boldsymbol{x}}^t - \mathrm{I}\right) d\boldsymbol{x} = \frac{1}{(n-1)}(\mathrm{R}-\mathrm{I})d\boldsymbol{x}, \tag{4}$$

$$dx_i = \frac{1}{(n-1)}\left[r_{i1}dx_1 + \cdots + r_{i,i-1}dx_{i-1} + r_{i,i+1}dx_{i+1} + \cdots + r_{in}dx_n\right]. \tag{5}$$

where a matrix $\mathbf{I}$ is the identity matrix. Notice that total differentiation of x_i is $dx_i = \partial x_i/\partial x_1 dx_1 + \cdots + \partial x_i/\partial x_{i-1} dx_{i-1} + \partial x_i/\partial x_{i+1} dx_{i+1} + \cdots + \partial x_i/\partial x_n dx_n$. We can represent the importances $\boldsymbol{x}$ as a system of total differential equations.

We obtain an algebraic representation of x_i by integrating the equation (5).

$$x_i = \int dx_i = \frac{1}{(n-1)}\left[\int r_{i1}\, dx_1 + \cdots + \int r_{in}\, dx_n\right]$$
$$= \frac{1}{(n-1)}\left[r_{i1}x_1 + \cdots + r_{i,i-1}x_{i-1} + r_{i,i+1}x_{i+1} + \cdots + r_{in}x_n\right] - d_i, \tag{6}$$

$$\boldsymbol{x} = \frac{1}{n-1}(\mathrm{R}-\mathrm{I})\boldsymbol{x} - \boldsymbol{d} \tag{7}$$

where $\boldsymbol{d} = [d_1, \cdots, d_n]^t$ is a constant of integration. To determine the constant, we reformulate the equation (7).

$$\mathrm{R}\boldsymbol{x} = n\boldsymbol{x} + (n-1)\boldsymbol{d}. \tag{8}$$

Let $\widehat{\boldsymbol{x}}$ be an eigenvector of an eigenvalue λ of R,

$$\mathrm{R}\widehat{\boldsymbol{x}} = \lambda\widehat{\boldsymbol{x}} = n\widehat{\boldsymbol{x}} + (n-1)\boldsymbol{d}, \tag{9}$$

$$\boldsymbol{d} = \frac{\lambda - n}{(n-1)}\widehat{\boldsymbol{x}}. \tag{10}$$

We obtain a representation of importances as an equation system:

$$\boldsymbol{x} = \frac{1}{n-1}(\mathrm{R}-\mathrm{I})\boldsymbol{x} = \frac{\lambda-n}{n-1}\widehat{\boldsymbol{x}}, \tag{11}$$

$$\frac{1}{n-1}(\mathrm{R}-n\mathrm{I})\boldsymbol{x} = \frac{\lambda-n}{n-1}\widehat{\boldsymbol{x}}. \tag{12}$$

If null-space of the matrix $(\mathrm{R}-n\mathrm{I})$ has some dimensions, then

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{x}. \tag{13}$$

A vector $\boldsymbol{y}$ is solutions of the equation $(\mathrm{R}-n\mathrm{I})\boldsymbol{y}$=0.

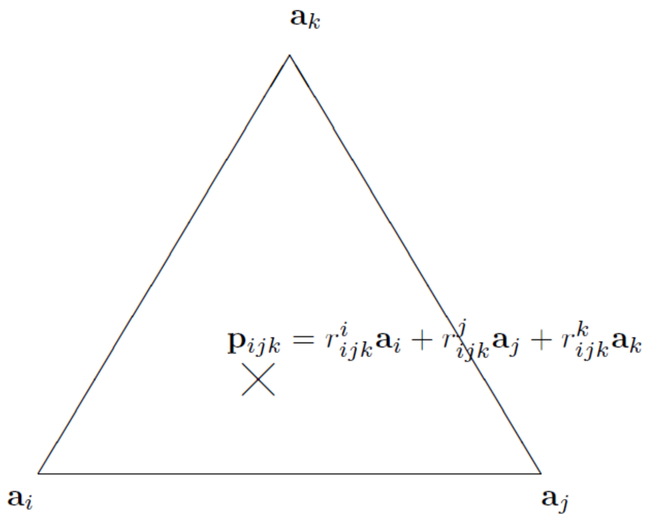

Figure 1 A ternary diagram for ternary comparison method

4. An analysis of the ternary comparison method

We can show importances of elements with indicating points on a triangle. This is based on that an interior point of triangle on Cartesian coordinates is expressed in weighted average of vertexes, and sum of the weights is 1.

For examples, let us consider a ternary diagram(Figure 1). The vertex $\boldsymbol{a}_i$ of the figure represents the element a_i. Decision makers put a point on the diagram, and let the point be $\boldsymbol{p}_{ijk}$. The point can be expressed in weighted average of vertex $\boldsymbol{a}_i$; $\boldsymbol{p}_{ijk} = r^i_{ijk}\boldsymbol{a}_i + r^j_{ijk}\boldsymbol{a}_j + r^k_{ijk}\boldsymbol{a}_k$ whose weights r^i_{ijk} can be determined uniquely, and $\sum_{\xi\in\{i,j,k\}} r^\xi_{ijk} = 1$. If there are only three elements, then their importances are determined at a glance; an importance of an element a_i is $x_i = \mathrm{r}^i_{\mathrm{ijk}}$. We refer to the method as *ternary comparison* in this paper.

For all three-piece sets of elements, decision makers indicate equilibrium points on their ternary diagram, and prepare values $r^{\mathrm{i}}_{\mathrm{ijk}}$, $i, j, k = 1, \cdots, n$. We revise the assumption of the relation of estimated values of ratio and importance x_i for the method as follows:

$$\frac{1}{r^i_{ijk}} \equiv \frac{\partial\left(x_i + x_j + x_k\right)}{\partial x_i}. \tag{14}$$

To find algebraic representations of importances x_i on the ternary comparison method, we integrate the equation (14):

$$\int \frac{1}{r^i_{ijk}}\, dx_i = \int \frac{\partial\left(x_i + x_j + x_k\right)}{\partial x_i}\, dx_i, \tag{15}$$

$$\frac{1}{r^i_{ijk}} x_i = x_i + x_j + x_k - d_i. \tag{16}$$

When we fix one element of triple to x_i, the number of triples which contains x_i is ${}_2\mathrm{C}_{n-1}$, the number of triples which contains x_j and x_i is ${}_1\mathrm{C}_{n-2} = n-2$. We add all equations whose left side has the term x_i:

$$s^i x_i = (n-2)x_1 + \cdots + (n-2)x_{i-1} + {}_2\mathrm{C}_{n-1} x_i + (n-2)x_{i+1} + \cdots + (n-2)x_n - d_i' \tag{17}$$

where s^i is sum of $1/r_{ijk}$ for all pairs $j,k \in \{1,\cdots,n\}\backslash\{i\}$. To determine d_i, which is constant of integration, we substitute 1 for x_i:

$$s^i = (n-1)(n-2) + {}_2\mathrm{C}_{n-1} x_i - d_i', \tag{18}$$

$$d_i'' = \frac{d_i'}{(n-1)(n-2)} = 1 - \frac{s^i - {}_2\mathrm{C}_{n-1}}{(n-1)(n-2)}. \tag{19}$$

And we obtain a equations system of the importance vector $\boldsymbol{x}$:

$$\mathrm{Q}\boldsymbol{x} = \boldsymbol{d}'', \quad \mathrm{Q} = (q_{ij}) = \begin{cases} (n-2) & i \neq j \\ ({}_2\mathrm{C}_{n-1} - s^i) & i = j \end{cases} \tag{20}$$

Elements of $\boldsymbol{d}''$ are in the equation (19). If null-space of the matrix Q has some dimensions, then we can write the importance vector $\boldsymbol{x}$ in

$$\boldsymbol{x} = \boldsymbol{y} + \mathbf{1}. \tag{21}$$

A vector $\boldsymbol{y}$ is a solutions of the equation $\mathrm{Q}\boldsymbol{y} = 0$.

5. Conclusions

We propose some algebraic representations for comparison methods of AHP. A key idea is that we regard ratios of importances as partial differentiations of them. Relations between importances are derived directly from these differentiations. In section 3, we also naturally introduce eigenvector and C.I., which is the term $(\lambda - n)/(n-1)$, into the representations of importances of the pairwise comparison method. The vector is a particular solution of the system of differential equations, and C.I. is a coefficient of nonhomogeneous term of the equations.

We also analyze the ternary comparison method. The method is a variation of visual analog scaling, and introduced clearly in AHP's fields at first, probably. We expect that we can construct computer interface for AHP with the method.

Estimated ratios of comparison methods can be regarded as differentials of importances without any fault. This means that we can apply comparison methods to machine learnings, or can retrieve importances of any element automatically. Because, in real societies, there are many numeric calculations of differentiations and many event are represented in their differentiations.

6. Key References

Harker, P. and Vargas, L. (1987). The theory of ratio scale estimation: Saaty's analytic hierarchy process. *Management Science, 33(11)*, 1383–1403.

Saaty, T. (1980). *The Analytic Hierarchy Process*, McGraw-Hill, New York.