Representations, Ratios, and Units

ABSTRACT

Between the numerator and denominator, there is a fine line. Only a fraction appreciates the distinction of what is above or below that line. The denominator and the "of what" is the unit in a pairwise comparison and in priority vectors. Pairwise comparisons provide the unit conversions between the elements being compared. These relationships are key to the problem definition and representation. As we understand what is above and below the fine line we come to recognize, appreciate, and respect the unit. In AHP/ANP it is important to recognize the nature of the problem and how units are used to represent it.

Keywords: pairwise comparisons, rank reversal, unit factor method, dimensional analysis

1. Introduction

Consider a simple paired comparison between two objects (alternatives, criteria, etc.) according to a certain property *p*. If object *A* contains 4 times more of *p* than object *B*, then the pairwise comparison A_p/B_p yields 4/1, or that A_p is 4 of 1 B_p unit.

We are able to express A_p as a quantity of B_p! With paired comparisons the object in the denominator is the relative unit of measurement. The pairwise comparison of B_p to A_p yields 1/4 of 1 A_p unit. For the "one" value in the denominator we have to ask ourselves "one of what unit"? The "of what unit?" is the object in the denominator. Different objects in the denominator represent different units.

Next, consider two objects where the ratios are 1/4 and 1/5. We know from elementary math that we cannot add them unless they are converted to a common denominator, such as 5/20 and 4/20. The common denominator creates a common counting unit that enables legitimate addition of the ratios. Some people may believe that $A_p/B_p + B_p/A_p = 4¼$ or 17/4. That is the case if you are counting with numbers and treating them as having no units. However, the two values are in different units; you would be adding things like apples and oranges. Even if we are adding just apples, ½ of 1 apple + ½ of 1 dozen apples does not equal 1 apple. Adding requires a common denominator and a common "of what unit" to be combined. As Zahir (2007) explained "that a 1 [of the unit] here does not necessarily equal 1[of the unit] there.' Zahir's statement applies to pairwise comparisons, to aggregating criteria clusters, and to different columns in a Supermatrix.

In the AHP, we calculate vectors of ratio priorities that sum to unity. A priority vector from the pairwise ratio matrix with elements A_p and B_p would typically have been normalized to [0.8, 0.2], summing to unity with A_p still 4 times larger than B_p. Normalization creates a new unit -- in this case, a composite object representing ($A_p + B_p$) in the denominator rather than a single object such as A_p or B_p.

Because the choice of unit is arbitrary, miscommunication and confusion can arise if different parties are referring to different units (Wedley & Choo, 2011: Zahir, 2007). This is particularly the case when aggregating across the units of different criteria Pairwise comparisons between criteria clusters are one way to provide the unit conversions to obtain an aggregate composite unit.

We explore that problem, the aggregation across the different units of criteria by revisiting the controversial rank reversal issues first identified by Belton and Gear (1983, 1985) We show that AHP's distributive mode and Belton and Gear's ideal procedure each provide correct answers for different problems with different units, unit conversions, and relationships. In other words, each is a different problem representation with its answer expressed in different units.

2. Conclusion

Once we understand that each problem has different units it becomes clear that they are different representations. "Rank reversal" and rank stability become natural and expected when there are different problems. In AHP/ANP it is important to recognize the nature of the problem and how the units are used to represent it. Being aware of units and different representations opens the door to answer new and longstanding questions in the AHP.

3. Key References

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Belton, V., & Gear, T. (1985). The legitimacy of rank reversal--A comment. *Omega*, *13*(3), 143-144.

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4. Appendices

Figure 1 presents data and analysis of the rank reversal issue that was first revealed by Belton and Gear (1983). We approach it from the perspective of units of measure that are expressed in the denominator of ratios. In the figure, the common data used for calculations is placed in the middle of the maxima and distributive modes of calculation. The synthesized results for both methods are displayed on the right. Regarding the data, Belton and Gear presented count data that they referred to as scores for each alternative on each criterion. We took the liberty to add Dollar, Euro and Yen symbols to those numbers to emphasize that the criteria are different properties.

Belton and Gear started with three alternatives (A, B & C) and later added alternative D that was a copy of alternative B. They exposed what they considered to be a fault with AHP's distributive mode. B was the preferred alternative when three alternatives were considered, but slipped to second place when D was added. Belton and Gear's proposed a maxima mode showed no such reversal.

Belton and Gear Maxima Mode
Maxima Object as unit

	C1=$	C2=€	C3=¥
Weights	1/3	1/3	1/3
A	1/9	9/9=1	8/9
B	9/9 =1	1/9	9/9=1
C	1/9	1/9	1/9
Sum=	11/9	11/9	18/9
Unit =	9$	9 €	9¥
D=B	9/9 =1	1/9	9/9=1
Same Unit =	9$	9 €	9¥

Data Representation
Three Alternatives

	C1	C2	C3
A	1$	9 €	8¥
B	9$	1 €	9¥
C	1$	1 €	1¥
Total	11$	11 €	18¥
Fourth Alternative added			
D=B	9$	1 €	9¥
New Total	20$	12 €	27¥

AHP Distributive Mode
Composite object as unit

	C1=$	C2=€	C3=¥
Weights:	1/3	1/3	1/3
A	1/11	9/11	8/18
B	9/11	1/11	9/18
C	1/11	1/11	1/18
Sum=	11/11=1	11/11=1	18/18=1
Unit =	11$	11 €	18¥
D=B	9/20	1/12	9/27
New Unit	20$	12 €	27¥

Synthesized Results

	Maxima 3	Maxima 4	Distributive 3	Distributive 4
A	18/27	18/27	268/594	148/405
B	19/27	19/27	279/594	117/405
C	3/27	3/27	47/594	23/405
D		19/27		117/405
	40/27	59/27	594/594	405/405

Figure 1 Data Representations from Belton and Gear, 1983

To appreciate the role played by units of measure, it is useful to understand how Belton and Gear considered the relationship between criteria weights and units of measure. In a revealing follow-up article, Belton and Gear (1985) explained that they interpretated criteria weights as scaling factors between the units of measure for the various criteria. This linking of criteria weights to the units being employed means that "… each criterion should be compared by direct reference to the options defining those units." Thus, the criteria weights represent the value of the units of each column and the relationships of unit between columns. On the left side of Figure 1, we can see that Belton and Gear chose the maxima alternatives as the units of measure (B for C1, A for C2 and B for C3).

Fortuitously and perhaps unfortunately, each of those alternatives chosen for determining criteria weights had a score of 9, Belton and Gear considered them equal and assigned equal weights of 1/3 to the three criteria. Had the units been 9$, 9 € and 9 ¥ rather than scores, we know the conversion factors would not be equal.
Note for the maxima mode that each denominator in criterion columns has the same count of 9. This signifies that those column vectors are assumed to be measured in reference to the same unit of measure. Subsequent multiplication by 1/3 during rescaling changes the denominator value to 27 (i.e. a different unit of measure). Notice that the new unit is the same across all criteria. This implies that the criteria weights converted the columns to commensurate units that could be directly added to a synthesized value. For the 3-alternative problem, B has is preferred (synthesized value of 19/27).

The fourth alternative added to the choice set did not have a higher maxima value on any of the criteria. Hence, there was no reason to change the reference unit and criteria weights. Being a copy of B, D has the same synthesized result as B. Since units of measure did not change, synthesized results remain the same and no rank reversal is observed in the results.

The distributive mode of AHP operates quite differently. Criteria weights are interpreted as weights in distributing an overall quantum unit to items listed lower in a hierarchy. As well, alternatives are expressed in relative values that sum to unity. The denominators and location of unit values are in Figure 1, this unit-sum requirement implies that a composite of all alternatives is the unit of measure. When D is added, it joins the other alternatives in establishing the composite unit for the columns. For example, the composite unit for C1 with 3 alternatives is 11$ but becomes 20$. That change in unit alters the proportion of criteria weight that is received by each alternative. When those different proportions are added up across criteria, the ratio between pre-existing alternatives changes. That change may or may not cause rank reversal. As shown in the synthesized results for the distributive mode, the change was sufficient to cause ranks to change. A with 4 alternatives becomes better than B which was dominant with 3 alternatives.

In AHP procedures, an ideal mode was implemented to avoid rank reversal. That mode is similar to the maxima procedure, but it does not anchor the criteria weights to the units of the columns. It is interesting to note that the distributive mode could have easily been adopted to avoid rank reversal – simply do not allow changes in unit if alternatives are added or deleted from the choice set. For example, if the composite units for 3 alternatives was maintained when D was added, then D could be measured in terms of that composite and no rank reversal would occur. Units matter. When we change units we are dealing in a different dimension.

The reader may be thinking one method is inferior to the other. We do not take that position. Both the maxima and the distributive modes are correct procedures for different representations. The distributive mode is the appropriate representation for allocation problems where some fixed quantum is to be distributed to divisions, recipients or beneficiaries that can vary in number. The maxima mode is the appropriate representation when benchmarks are needed to measure alternatives against fixed standards. Different representations can use units in different manners.