Contents

ANP Row Sensitivity's Natural Fixed Point

Bill Adams

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Abstract

In ANP row sensitivity as defined in [**?**] the standard family of row perturbations $F_{W,r,p_0}(p)$ has an arbitrary choice of fixed point p_0. Which value one chooses here changes the results of the various calculations quantitatively (if not qualitatively). There is, however, a natural choice for the fixed point. This natural choice makes the graph of a given alternative differentiable. This gives, for instance, a unique marginal influence (see [**?**]) value (instead of upper and lower values).

1 Introduction

ANP row sensitivity seeks to mimic the behavior of AHP tree sensitivity (where one changes the local priority of a node and re-synthesizes, to arrive at new values for the alternatives). In software this is typically done via a user interface where one can grab a bar representing the local priority of the given node and drag it out longer or shorter (and the rest of it's siblings' priorities change accordingly) and one can see the alternative scores change.

Nothing in the previous statement is unique to trees, and thus this algorithm could be applied to ANP networks. In fact it has been in the software SuperDecisions version 1 (in a more primitive user interface than the one described above). However, very often one gets very little sensitivity as a result of these calculations. This is because one is only changing the local priority of the given node. In trees a node only ever has one local priority. In networks a node may have many local priorities. As a result of this, and feedback within the network, often times one never sees any useful sensitivity from this calculation.

ANP row sensitivity as defined in [**?**] remedies this by changing the entire row of the supermatrix corresponding to the given node (rather than a single entry in the supermatrix, which is what the above corresponds to). In so doing we must be careful how we perturb the supermatrix, so as to maintain as much of the original ANP structure as possible. As a result of those rules we are left with essentially one parameter p_0, called the fixed value of the family. This parameter value corresponds to leaving the supermatrix unchanged. We are free to choose any value for this between 0 and 1 (exclusive).

The question naturally arises, is there a natural value of p_0 to choose? Setting $p = 0.5$ has a charm to it, in that for values above 0.5 the node's priority goes up and below 0.5 the value goes downward. However using $p_0 = 0.5$ does not correspond well with the tree analogy (translating the tree analogy to this terminology, the standard calculation used is equivalent to having p_0 equal to the local priority). We do not have a single local priority to use (in the ANP case), but we could use the average of the local priorities, or the global priority perhaps. All of these have a failing that the graphs of the alternatives scores with respect to the parameter p will not in general be differentiable at p_0 for these choices (in the tree analogy we do have differentiability everywhere). The question is, is there a choice of p_0 that gives differentiability of the alternative score functions?

In this paper we find that there is not a single unique value for p_0 with this behavior (for all alternatives). However, there are fixed point values $p_0 = \rho_{0,i}$ that make the graph of alternative i differentiable at p_0 (up to a minor assumption on the graph of the alternative score). Before we get to this result, let us quickly review ANP row sensitivity, and see an example of on the non-differentiability issue.

1.1 ANP Rows Sensitivity Review

The following is a brief review of the concepts involved in [**?**]. The purpose of ANP row sensitivity is to change all of the numerical information for a given node in a way that is consistent with the ANP structure, and recalculate the alternative values (much as tree sensitivity works). We do this by having a single parameter p that is between zero and one, which represents the importance of the given node. There is a parameter value p_0 (called the fixed point) which represents returning the node values to the original weights. For parameter values larger than p_0 the importance of the node goes up, and for parameter values less than p_0 the importance of the node goes down. Once the parameter is set, this updates values in the weighted supermatrix (although it can also be done with the unscaled supermatrix, working by clusters instead) and resynthesizes.

As can be seen from [**?**], there is essentially one way to do this calculation and preserve the ANP structure of the model. In the notation of that paper, let W be the weighted supermatrix of a single level of our model, ANP row sensitivity constructs a family of row perturbations of W. A family of row perturbations of W is a mapping $f : [0,1] \to M_{n,n}([0,1])$ that gives a weighted supermatrix $f(p)$ for each parameter value $p \in [0,1]$. This mapping must preserve the ANP structure of our original supermatrix. The only real choice is what to make our fixed point p_0. Once we have chosen that, the standard formula for the family of row perturbations of row r of W preserving the ANP structure is labeled $F_{W,r,p_0} : [0,1] \to M_{n,n}([0,1])$ and is defined in the following way.

1. Leave trivial columns unchanged. A trivial column is either a zero column, or a column with all zeroes except one entry that is one.

2. If $0 \leq p \leq p_0$ define $F_{W,r,p_0}(p)$ by scaling the r^{th} row by $\frac{p}{p_0}$ and scaling the other entries in the columns so as to keep the matrix stochastic.

3. If $p_0 \leq p \leq 1$ define $F_{W,r,p_0}(p)$ by leaving alone columns of W for which $W_{r,i} = 0$ and scaling all entries in the other columns, except for the entry in the r^{th} row, by $\frac{1-p}{1-p_0}$ (and change the entry in that r^{th} row so as to keep the matrix stochastic).

1.2 An Example Showing Non-differentiability

We use the 4node2 example model from [?] as our first example. It is a model with 2 clusters, one criteria cluster with nodes A and B, and the alternatives cluster with alternatives 1 and 2. The following is a table of alternative scores at various parameter values with $p_0 = 0.5$ doing row sensitivity of the first row.

p	alt 1	alt 2
0.500000	0.388144	0.611856
0.000100	0.251839	0.748161
0.050090	0.266465	0.733535
0.100080	0.280862	0.719138
0.150070	0.295034	0.704966
0.200060	0.308983	0.691017
0.250050	0.322712	0.677288
0.300040	0.336223	0.663777
0.350030	0.349520	0.650480
0.400020	0.362604	0.637396
0.450010	0.375478	0.624522
0.500000	0.388144	0.611856
0.549990	0.451119	0.548881
0.599980	0.507641	0.492359
0.649970	0.558482	0.441518
0.699960	0.604297	0.395703
0.749950	0.645645	0.354355
0.799940	0.683001	0.316999
0.849930	0.716776	0.283224
0.899920	0.747327	0.252673
0.949910	0.774962	0.225038

The approximation for the upper and lower derivatives of alternative 1 follows below.

$$upper = \frac{.451119 - .388144}{.549990 - .5} = 1.25975$$

$$lower = \frac{.375478 - .388144}{.450010 - .5} = 0.25337$$

Further calculations show this approximation to accurate, and thus the upper and lower derivatives are not equal. Therefore the alternative score is not differentiable with respect to the parameter p at $p = 0.5$. (Note: this is for $p_0 = 0.5$.)

If we use the global priority of 0.255563 as our p_0 value we get the upper and lower derivatives of the score of alternative 1 being closer to equal. The upper and lower derivatives at the fixed value are:

$$upper = 0.49$$

$$lower = 0.42$$

Thus, they are still not equal.

Instead, we could try using p_0 equal to the average of the row in the supermatrix. For the first row that average is

$$average = (0.37500 + 0.20001 + 0.04999 + 0.33333)/4 = 0.2395825$$

If we use $p_0 = 0.2395825$ and calculate upper and lower derivatives of alt 1's score at p_0 we find

$$upper = 0.48$$

$$lower = 0.45$$

Thus again, they are not equal, and the score for alt 1 is not differentiable with respect to the parameter p at p_0.

In conclusion, none of the standard ideas for p_0 values gives alternative scores that are differentiable with respect to p at the fixed value p_0. That derivative has an important interpretation as the marginal influence, and in the current situation we have marginal influence broken up into upper and lower marginal influence. It would be advantageous if we could choose a value of p_0 at which upper and lower derivatives agree, so that we could have a single marginal influence value.

In addition, the lack of differentiability of the alternative scores at p_0 make the graphs of alternative scores with respect to p have a broken appearance at p_0, which is undesirable. An image of a graph of alt 1 and alt 2's score with respect to p with $p_0 = 0.5$ can be found in Figure **??** and one can see the break in the graphs at p_0.

The remainder of this paper addresses the problem of finding a p_0 that fixes this issue.

2 Results

In this section we define the *natural fixed point* for a family of ANP row perturbations. This natural fixed point removes the objections raised earlier if it exists. In addition we derive a formula for the natural fixed point.

In order to fix our terminology, let A be an ANP network with a alternatives, n total nodes, and with weighted supermatrix W. In addition $F_{W,r,p_0}(p)$ be the standard family of row perturbations of W in the r^{th} row with fixed point p_0.

SmartP0Img1.png

Figure 1: Graph with $p_0 = 0.5$

2.1 Definition and Concepts

The fundamental definition of this paper is that of the natural fixed point, which follows. [Natural Fixed Point of alternative i] With W, a, n, and $F_{W,r,p_0}(p)$ as defined above, we define the **natural fixed point for alternative i** to be the fixed point value that makes the function $a_i(p) =$"the score of alternative i with parameter p" differentiable at $p = p_0$. We denote it by $\rho_{0,i}$. We would like to note a few things immediately about this definition. From the definition it is not at all apparent that there should be only one such value. However, we are dealing with the standard family F_{W,r,p_0}, and we can readily understand how that family changes as we change p_0. It comes out from that there is at most one such value. At the moment we have no guarantee that there will always exist a $\rho_{0,i}$. If the graph of $a_i(p)$ has certain properties $\rho_{0,i}$ will exist, if the graph does not have those properties $\rho_{0,i}$ will not (a nice exclusion). Perhaps it is the case that all $a_i(p)$ graphs will have that property, but we do not have a proof of that. It is almost never the case that $\rho_{0,i}$ and $\rho_{0,j}$ are the same (with $i \neq j$ of course). The BigBurger example we calculate later is one such example where they are not equal. Notice that $\rho_{0,i}$ actually depends upon the row r we are doing sensitivity upon. When we wish to be explicit about noting the row we write $\rho_{0,r,i}$.

2.2 Main Results

The following is the main calculation result we have. With W, a, n, and $F_{W,r,p_0}(p)$ as defined above, with p_0 the fixed point of the family, we can calculate the natural fixed point for the family via the following formula (assuming it exists).

$$\rho_{0,i} = \frac{\beta}{\alpha + \beta}$$

where $\alpha = (1 - p_0)a'_{i+}(p_0)$ and $\beta = p_0 a'_{i-}(p_0)$ where $a'_{i+}(p_0)$ and $a'_{i-}(p_0)$ are the upper and lower derivative of the score of alternative i with respect to p where we use the original p_0 for the family and evaluate the derivatives at that fixed point.

Proof. Assume at least one $\rho_{0,i}$ exists. We have two different families of ANP row perturbations, $F_{W,r,p_0}(p)$ and $F_{W,r,\rho_{0,i}}(p)$. Based on the formula of $F_{W,r,p_0}(p)$ in [**?**] we can write that

$$F_{W,r,\rho_{0,i}}(p) = F_{W,r,p_0}(\gamma_{\rho_{0,i}}(p))$$

where $\gamma_{\rho_{0,i}} : [0, 1] \to [0, 1]$ is essentially a change of parameters. We can write a similar formula for the score of alternative i. Let us make our notation for the score of alternative i a bit more specific. Let

$$a_{p_0,i}(p)$$

be the score of alternative i when our fixed point is p_0. With this we can write

$$a_{\rho_{0,i},i}(p) = a_{p_0,i}(\gamma_{\rho_{0,i}}(p)).$$

This is the same $\gamma_{\rho_{0,i}}$ as defined above. And the formula for $\gamma_{\rho_{0,i}}$ follows directly from the definition of the family $F_{W,r,p_0}(p)$. Namely it is

$$\gamma_{\rho_{0,i}}(p) = \begin{cases} p \cdot \frac{p_0}{\rho_{0,i}} & \text{if } 0 \leq p \leq \rho_{0,i} \\ (p-1)\frac{1-p_0}{1-\rho_{0,i}} + 1 & \text{if } \rho_{0,i} < p \leq 1 \end{cases}$$

Notice the $\gamma_{\rho_{0,i}}(p)$ is a piecewise linear function. Using this function we can calculate the upper and lower derivatives of $a_{\rho_{0,i},i}(p)$ with respect to p at $p = \rho_{0,i}$. First we handle the upper derivative.

$$\begin{aligned} a'_{\rho_{0,i},i+}(\rho_{0,i}) &= \left.\frac{d}{dp_+}\right|_{\rho_{0,i}} a_{p_0,i}(\gamma_{\rho_{0,i}}(p)) \\ &= a'_{p_0,i+}(\gamma_{\rho_{0,i}}(\rho_{0,i})) \cdot \gamma'_{\rho_{0,i}+}(\rho_{0,i}) \\ &= a'_{p_0,i+}(p_0) \cdot \frac{1-p_0}{1-\rho_{0,i}} \end{aligned}$$

Now the lower derivative.

$$\begin{aligned} a'_{\rho_{0,i},i-}(\rho_{0,i}) &= \left.\frac{d}{dp_-}\right|_{\rho_{0,i}} a_{p_0,i}(\gamma_{\rho_{0,i}}(p)) \\ &= a'_{p_0,i-}(\gamma_{\rho_{0,i}}(\rho_{0,i})) \cdot \gamma'_{\rho_{0,i}-}(\rho_{0,i}) \\ &= a'_{p_0,i+}(p_0) \cdot \frac{p_0}{\rho_{0,i}} \end{aligned}$$

Now, by definition of $\rho_{0,i}$ we have the upper and lower derivatives should be equal. Before we do this let us setup the notation $a_+ = a'_{p_0,i+}(p_0)$ and $a_- = a'_{p_0,i-}(p_0)$ to make our equations a bit easier to read. We thus get the following sequence of equalities.

$$\begin{aligned} a'_{\rho_{0,i},i+}(\rho_{0,i}) &= a'_{\rho_{0,i},i-}(\rho_{0,i}) \\ a'_{p_0,i+}(p_0) \cdot \frac{1-p_0}{1-\rho_{0,i}} &= a'_{p_0,i+}(p_0) \cdot \frac{p_0}{\rho_{0,i}} \\ a_+ \cdot \frac{1-p_0}{1-\rho_{0,i}} &= a_- \cdot \frac{p_0}{\rho_{0,i}} \end{aligned}$$

Next, to simplify our equation further let $\alpha = a_+ \cdot (1-p_0)$ and $\beta = a_- \cdot p_0$. We can then continue the above equations as follows, and solve for $\rho_{0,i}$.

$$\begin{aligned}
a_+ \cdot \frac{1-p_0}{1-\rho_{0,i}} &= a_- \cdot \frac{p_0}{\rho_{0,i}} \\
\frac{\alpha}{1-\rho_{0,i}} &= \frac{\beta}{\rho_{0,i}} \\
\alpha\rho_{0,i} &= \beta(1-\rho_{0,i}) \\
\alpha\rho_{0,i} &= \beta - \beta\rho_{0,i} \\
\alpha\rho_{0,i} + \beta\rho_{0,i} &= \beta \\
\rho_{0,i}(\alpha+\beta) &= \beta \\
\rho_{0,i} &= \frac{\beta}{\alpha+\beta}
\end{aligned}$$

Which completes the proof. □

This equation uniquely defines any $\rho_{0,i}$, thus there is at most one such natural fixed point. The next question is when $\rho_{0,i}$ exists. Based upon the formula for $\rho_{0,i}$ we can easily find the following corollary. With notation as in the previous theorem $\rho_{0,i}$ exists iff a_- and a_+ share the same sign.

Proof. From the previous theorem we have that any $\rho_{0,i}$ equaling $\frac{\beta}{\alpha+\beta}$ will be the natural fixed point. Thus for the natural fixed point to exist that formula must give us a value between 0 and 1. So we have

$$\begin{array}{rcl}
0 < & \rho_{0,i} & < 1 \\
0 < & \frac{\beta}{\alpha+\beta} & < 1
\end{array}$$

We can continue to work on this by separating into cases, where the denominator is positive or negative. If the denominator is positive we can continue as follows.

$$\begin{array}{rcl}
0 < & \frac{\beta}{\alpha+\beta} & < 1 \\
0 < & \beta & < \alpha+\beta \\
-\beta < & 0 & < \alpha \\
-a_-p_0 < & 0 & < a_+(1-p_0)
\end{array}$$

The left hand side of the last inequality says $a_- > 0$. The right hand side of the last inequality says $a_+ > 0$. So they share the same sign, and we continue.

$$\begin{array}{rcl}
-a_-p_0 < & 0 & < a_+(1-p_0) \\
\frac{-p_0}{1-p_0} < & 0 & < \frac{a_+}{a_-}
\end{array}$$

Notice the left hand hand side of this inequality is always true, and since both a_+ and a are positive the right hand side is trivially true. Thus if $\alpha+\beta > 0$ both a_+ and a_- need only be positive (and the opposite as well, if a_+ and a_- are both positive we can work backwards to get $\alpha+\beta > 0$ and that $\rho_{0,i}$ exists.)

Working with $\alpha+\beta < 0$ works similarly, giving a_- and a_+ are both negative. □

As a second corollary simplifies our formulas in a particular case of p_0. With notation as in theorem **??** and $p_0 = 0.5$ we have the following formula:

$$\rho_{0,i} = \frac{a_-}{a_+ + a_-}$$

Proof. Merely plugin $p_0 = 0.5$ to the formulas for α and β to arrive at the result. $\square$

3 Calculations and Applications

In this section we will calculate the natural fixed points by hand for two models, using SuperDecisions to facilitate the calculation. We end this section by discussing some applications of natural fixed points.

In order to calculate $\rho_{0,i}$ we will use the formula from theorem [**?**] to perform the calculation. In order to do this, we need to know a_+ and a_-, that is the upper and lower derivatives for some value p_0. We use SuperDecisions to do the heavy lifting for this (other examples using SuperDecisions to calculate upper and lower derivatives is given in [**?**]).

3.1 4node2 model

This model is the model as used in the examples of [**?**]. It is a model with two clusters (a criteria cluster and alternatives cluster) each of which contain two nodes (two criteria A and B and two alternatives 1 and 2). All nodes are connected to one another with pairwise comparison data inputted. The resulting weighted supermatrix is the following (the order of the nodes are A, B, 1, and finally 2).

$$W = \begin{bmatrix} 0.37500 & 0.20001 & 0.04999 & 0.33333 \\ 0.12500 & 0.29999 & 0.45000 & 0.16667 \\ 0.33333 & 0.04999 & 0.27500 & 0.15001 \\ 0.16667 & 0.45000 & 0.22500 & 0.34999 \end{bmatrix}$$

The calculation of the upper and lower of derivatives using $p_0 = 0.5$ gives the following results.

	D(Normal) 1	D(Normal) 2
Original	0.39	0.61
A:upper	0.73	-0.73
B:upper	-0.68	0.68
1:upper	2.45	-2.45
2:upper	-1.66	1.66
A:lower	0.22	-0.22
B:lower	-0.2	0.2
1:lower	0.58	-0.58
2:lower	-0.67	0.67

In order to calculate $\rho_{0,1,1}$ (that is the natural fixed point for row 1, corresponding to node A, for alternative 1, which is in fact named node 1) we have $a_+ = A : upper$ and $a_- = A : lower$ from the given table. Thus we get from corollary **??**

$$\rho_{0,1,1} = \frac{a_-}{a_+ + a_-} = 0.22/0.95 = 0.23$$

Similarly we get the following

$$\begin{aligned}\rho_{0,2,1} &= 0.228999\\ \rho_{0,3,1} &= 0.190516\\ \rho_{0,4,1} &= 0.287952\end{aligned}$$

Because there are only two alternatives it turns out $\rho_{0,r,1} = \rho_{0,r,2}$ for all rows r. In our next example we will see that need not occur if there are more than two alternatives.

3.2 BigBurger model

This is one of the standard models included in SuperDecisions. We will calculate the natural fixed point for alternatives 1, 2, 3 for sensitivity with row for 1 Subs.

The resulting upper and lower derivatives are the following.

Row	Deriv Mac	Deriv BK	Deriv W
1 Subs:upper	-0.462106	0.141492	0.32061
1 Subs:lower	-0.008785	0.002465	0.00632

Notice that node 1 Subs is row 24 of the original supermatrix. We will calculate $\rho_{0,24,1}$ the natural fixed point for row 24 alternative 1 (namely MacDonalds). In that case $a_+ = -0.462106$ and $a_- = -0.008785$, and from corollary **??** we have the following.

$$\rho_{0,24,1} = \frac{a_-}{a_+ + a_-} = -0.008785/(-0.462106 + -0.008785) = .01865612$$

Similarly we find the following for each alternative.

$$\begin{aligned}\rho_{0,24,1} &= 0.018656\\ \rho_{0,24,2} &= 0.017147\\ \rho_{0,24,3} &= 0.019350\end{aligned}$$

Notice that the natural fixed points for each alternative are different in this case.

3.3 Applications

There are two specific applications we discuss here. One has to do with analysis of nodes of interest where one should further analyze one's judgments. The other has to do with the standard sensitivity graph.

In the first application, consider the process of entering comparison data (or multiple users entering comparison data). This can be a tedious process, especially to get data as accurate as possible. However, we can imagine going through the pairwise comparisons first as a rough draft, not debating over a 4 versus a 5, and synthesizing. If we then compute total marginal influence using the natural fixed point we get a single marginal influence value for each row and each alternative. If we do a euclidean size of the derivative vector for each row (summing over the alternatives) we get a total marginal influence for the given row. If we then sort the rows on this total marginal influence we now have a measure of which nodes are most sensitive to small changes. That is the nodes we should focus in on for further refining our original (approximate) judgments. Typically there are very few of these nodes. In addition we know that when the total marginal influence of a row is very close to zero there is no use refining the pairwise comparisons for that node (as they make essentially no difference). This process could dramatically increase the speed of inputing data without sacrificing accuracy.

The second application has to do with graphing the alternative scores as a function of the parameter p. As we can see in the figure on page **??**, there are breaks in the graph over p_0 unless we choose p_0 to be the natural fixed point. If we choose p_0 for each graph to be the natural fixed point, the graphs loose the breaks. With this change the graphs now appear as they would in the case of AHP trees. In addition this process precisely reproduces the AHP graph when our network happens to be a tree.