TWO TECHNIQUES TO MODIFY JUDGEMENTS MATRICES

Da Ma and Bo Guo
Department of Systems engineering and Mathematics
Changsha Institute of Technology
Changsha, China

ABSTRACT

It is very difficult to revise A to satisfy consistency if the order of the judgement matrix A is higher. A relative difference function is proposed in this paper and by means of the function we present two techniques to test individual element's consistency to find out most possibly wrong elements if A doesn't satisfy the consistency index and the reference criteria of modification are given. Under the guidance of the two methods it is very easy and efficient to modify A and much labor can be saved.

1. INTRODUCTION

In the Analytic Hierarchy Process (AHP) the final rank of decision elements depends on the matrix A of pairwise comparisons. An element a_{ij} in A contains the preference grade of decision makers between decision elements i and j. Due to the nondeterminacy, fuzziness of matter, and the fuzziness of people's thought, the inconsistency of a judgement matrix is inevitable. But we can put up with the inconsistency in a certain range. Therefore a criterion to check the consistency of judgements matrices is given in AHP. The pairwise comparisons are regarded to be contradictory if the observed consistency index exceeds the range. The judgement matrix should be revised to satisfy the consistency index.

When the order of A is high, generally the initial comparisons matrix A is inconsistent. A matrix of n order contains $n(n-1)/2$ times of pairwise comparisons. It is very difficult to find out where the errors are located if there is no criterion of modification and revisiong A. It's possible to introduce new contradictions, which make it difficult for one to practise AHP. Hence besides testing the consistency of matrix A and modifying A, we should have a method to test individual elements of A in order to find out where the errors most probably occur when A doesn't satisfy consistency.

Based on the above ssatements, this paper proposes two methods to revise A when judgement matrix A is inconsistent. The methods can be used to find out elements of A that are most probably wrong, and reference criteria of revision are given. It's easy and efficient to modify A and much labor can be saved by means of the two methods.

2. THE RELATIVE DIFFERENCE FUNCTION

In AHP the intensity of relative importance of alternatives are represented in nine scores system. When the importance of alternative i is larger than that of j, the increment of importance grades is uniformly expressed by the increment of scores. The scores corresponding to comparison grades are nonuniformly compressed into interval (0,1) if the importance of i is less than that of j. We form a relative difference function which transforms the nonuniform scores of pairwise comparisons into uniform relative differences. The function is as follows.

$$f(x)=\begin{cases} x-1 & x\geq 1, \\ 1-1/x & x<1 \end{cases}$$

Where argument x takes the values of the scores of pairwise comparisons. The uniformity and meanings of function f measures are stated in table 1.

3. METHODS TO TEST ELEMENTS CONSISTENCY

We suppose that most judgements in alternatives are correct. There are only a few mistakes, which can be recognized under some hints, and the deviation of the whole consistency index C.R. is not large.

table 1 Relative different measures and meanings

f measures	Definition	Intensity of Relative Importance
-8	Extreme difference	1/9
-6	Demonstrated difference	1/7
-4	Strong difference	1/5
-2	Slight difference	1/3
0	No difference	1
2	Slight difference	3
4	Strong difference	5
6	Demonstrated difference	7
8	Extreme difference	9
1,3,5,7	Intermediate values between	2,4,6,8
-1,-3,-5,-7	the two adjacent judgements	1/2,1/4,1/6,1/8

In comparisons matrix $A=(a_{ij})$ is a relative weight by direct judgement between alternatives i and j. Much indirect information between i and j is also included in A such as a_{ik}/a_{jk}, which is a secondhand judgement through alternative k. If A is a consistent matrix

$$a_{ij}=a_{ik}/a_{jk} \qquad \text{for all } i,\ j,\ k,\ =1,2,\ \ldots,n.$$

According to the assumption that most judgements are correct, we should have been able to estimate a_{ij} by $\sum a_{ik}/a_{jk}/(n-1)$, but for the nonuniformity of judgement criteria a_{ij} can not be well estimated by the sum formula. After the relative difference function is introduced, the terms of equal deviation grades of relative weights play the same important role in the sum formula. Let

$$SA(i,j)=\frac{1}{n-1}\sum_{\substack{k=1\\k\neq j}}^{n} f(a_{ik}/a_{jk}),$$

$$Ml(i,j)=f(a_{ij}) - SA(i,j)$$

$Ml(i,j)$ is applied to examine whether a_{ij} is consistent with most comparisons or not. the matrix $Ml=(ml(i,j))$ has the following properties.

(1) Anti-symmetric relation: $Ml=-Ml$

(2) Central property: $Ml=0$, if A is consistent

(3) Uniformity: The difference grade between a_{ij} and $\frac{1}{n-1}\sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}/a_{jk}$

(4) Directivity: a_{ij} is larger if $M1(i,j)>0$

a_{ij} is less if $M1(i,j)<0$.

a_{ij} is proper if $M1(i,j)=0$

Let the right eigenvector is $W=(w_1,\ldots, w_n)^T$ corresponding to the principal eigenvalue of A. Then w_i/w_j is a synthesis of relative weights between alternatives i and j. $a_{ij}=w_i/w_j$ if A is consistent. The consistency of A can be improved when a_{ij} approaches w_i/w_j for $B=(w_i/w_j)_{nxn}$ is a matrix of consistency, which can also be seen in the expression of consistency index

$$C.I. = \frac{\lambda_{max} - n}{n-1} = -1 + (1/\ n(n-1)) \sum_{\substack{i=1 \\ j\neq i}}^{n} a_{ij} w_j/w_i$$

The closer a_{ij} is to w_i/w_j, i.e. C.I. is closer to zero, the more consistent is A. So

$$M2(i,j)=f(a_{ij})-f(w_i/w_j)$$

can be used to test if a_{ij} is consistent with most judgements. $M2=(M2(i,j,))_{nxn}$ possesses the same four properties that have been stated above. By means of M1 or M2 the pairwise comparisons which are most possibly wrong can be found if A doesn't satisfy consistency. The larger the $|M1(i, j)|$ or $|M2(i, j)|$ is, the greater the error possibility of a_{ij} is. The consistency between individual judgements and the whole judgements is tested by M1 or M2.

4. METHODS OF MODIFICATION

If A doesn't satisfy the consistency index, M1 or M2 is calculated. Some judgements are reassessed corresponding to the elements of M1 or M2 which absolute values are larger. The consistency of A is tested once again when A is revised. This process isn't done until a satisfies the consistency index. It is likely to be better that M1 and M2 are used together. a_{ij} is revised if both $|M1(i,j)|$ and elements are moditied at a time, and of course the reciprocals of theirs have to be accordingly revised too.

EXAMPLE 1 Let A be a judgement matrix.

$$A = \begin{bmatrix} 1 & 5 & 1/7 & 1/5 & 1 & 1/5 \\ 1/5 & 1 & 1/5 & 1/3 & 1 & 1/3 \\ 7 & 5 & 1 & 5 & 5 & 1 \\ 5 & 3 & 1/5 & 1 & 5 & 1/3 \\ 1 & 1 & 1/5 & 1/5 & 1 & 1/7 \\ 5 & 3 & 1 & 3 & 7 & 1 \end{bmatrix}$$

By computation C.R.=0.128>0.1, i.e. A in incons[illegible] which needs revision Then M1 and M2 are obtained.

$$M1 = \begin{bmatrix} 0.0 & 3.5 & -1.6 & -2.3 & -0.8 & 1.9 \\ -3.5 & 0.0 & 7.6 & 4.0 & 0.4 & 6.8 \\ -1.6 & -7.6 & 0.0 & 2.6 & -4.8 & -0.3 \\ 2.3 & -4.0 & -2.6 & 0.0 & 1.7 & -0.7 \\ 0.8 & -0.4 & 4.8 & -1.7 & 0.0 & 0.0 \\ -1.9 & -6.8 & 0.3 & 0.7 & 0.0 & 0.0 \end{bmatrix}$$

$$M2 = \begin{bmatrix} 0.0 & 3.5 & -2.2 & -2.9 & -0.6 & -1.1 \\ -3.5 & 0.0 & 2.4 & 0.3 & -0.1 & 2.9 \\ 2.2 & -2.4 & 0.0 & 2.7 & -2.8 & -0.2 \\ 2.9 & -0.3 & -2.7 & 0.0 & 1.6 & -1.2 \\ 0.6 & 0.1 & 2.8 & -1.6 & 0.0 & -0.7 \\ 1.1 & -2.9 & 0.2 & 1.2 & 0.7 & 0.0 \end{bmatrix}$$

We use M1 and M2 together. By comparision, $a_{12}, a_{32}, a_{62}, a_{35}$ are elected to be modified at a time to be $a_{12}=4$, $a_{32}=6$, $a_{62}=4$, $a_{35}=6$. Their reciprocals are accordingly changed too. Then the revised A is tested on consistency. C.R.=0.090< 0.1, which showes that the revised A is of acceptable consistency.

On the assumption that

$$f(a_{ij})-SA(i,j)=0$$

$f(a_{ij})$ gets an error Δ if a_{ij} has a disturbance ε_{ij}.

$$|M1(i,j)|=\left|f(a_{ij})+\Delta-SA(i,j)-\frac{1}{n-1}\Delta\right|=\frac{n-2}{n-1}|\Delta|,$$

from which table 2 is obtained, which can be extended by taking $|\Delta|=4, 5, \ldots$

In accordance with the order n of A and the value of $|M1(i,j)|$, $|\Delta|$ can be consulted in table 2 Then ε_{ij} corresponding to $|\Delta|$ can be deduced. ε_{ij} can be taken as a reference value to revise a_{ij}. Conversely if we want to control $|\Delta|$ under a certain level the correspondeng $|M1(i,j)|$ can be taken as a value for reference too.

Large numbers of statistical experiments indecate that there is a statistical correlation between M1 and M2, and it is as follows.

$$M2 = 0.5\ M1$$
$$r= 0.8 > 0.42 \quad (\alpha=0.01),$$

where r is the correlation coefficient. From this the reference table of M2 for revision can be obtained.

Table 2 Reference values of M1 for revision

$|M1(i,j)|$ \ $|\Delta|$ / n	1	2	3
4	0.67	1.33	2.00
5	0.75	1.50	2.25
6	0.80	1.60	2.40
7	0.83	1.67	2.50
8	0.86	1.71	2.57
9	0.88	1.75	2.63
10	0.89	1.78	2.67
11	0.90	1.80	2.70
12	0.91	1.82	2.73
13	0.92	1.83	2.75
14	0.92	1.85	2.77
15	0.93	1.86	2.79

5. CONCLUSION

Large numbers of practices indicate that the methods proposed here are simple and effecient when the order of a judgement matrix is higher without strict mathematical proofs. The proposed idea of the relative difference function that transforms the non-even nine scores system [1/9, 9] into even scores in [-8, 8] may find application in the test of the consistency of judgement matrice.

REFERENCES

SAATY, T.L. (1986), "Axiomatic Foundation of the Analytic Hierarchy Process", Management Science, Vol.32 No.7, P.851-855

ZAHEDI, F. (1986), "The Analytic Hierarchy Process A Survey of the Method and its Applications", Interfaces 16:4 , pp 96-108

SUBO XU, (1988), "The principle of the Analytic Hierarchy Process" Tianjin University press, (In Chinese)