MODELLING INCONSISTENCY IN THE AHP THROUGH CHOQUET INTEGRATION

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ABSTRACT

We propose an extension of Saaty's AHP based on Choquet integration. In our model an appropriate measure of inconsistency is explicitly considered in the aggregation process in order to attenuate (resp. emphasize) the priority values of the criteria with higher (resp. lower) average inconsistency with the remaining criteria.

Keywords: AHP, Inconsistency, Capacity, Choquet integral, and Shapley values.

1. Introduction

The Analytic Hierarchy Process (AHP), introduced by Thomas L. Saaty (1977, 1986, 1988; Saaty and Vargas, 1991), is a well-known multicriteria aggregation model based on pairwise comparison matrices at two fundamental levels: the lower level encodes pairwise comparison matrices between alternatives (one such matrix for each criterion) and the higher level encodes a single pairwise comparison matrix between criteria. In its most general form, the higher level of the AHP can be structured hierarchically, with several layers of criteria, but in this paper we focus on the single layer case, with a single matrix of pairwise comparisons between criteria.

The AHP extracts from each pairwise comparison matrix a vector of priority weights corresponding to the principal eigenvector or, alternatively, to the geometric mean vector. In both cases the priority vector has positive components normalized to unit sum. In this paper we consider only the geometric mean method, because its structural properties are more suited for our study. Once the priority vectors associated to the various pairwise comparison matrices are obtained, the AHP uses the priority vector at the higher level to aggregate (by means of weighted averaging) the lower level priority vectors.

More recently, Saaty proposed the Analytic Network Process (ANP) (1996, 2004a, 2004b; Saaty and Vargas, 2006) in order to incorporate in the AHP the effect of dependence and feedback in the structure of the model, within and between levels. In this paper, instead, we focus on the question of inconsistency and how it can be used to modulate the priority values of the various criteria.

Pairwise comparison matrices are typically inconsistent. In fact, the AHP does not require the decision maker to be consistent, but it might be relevant to estimate his/her degree of inconsistency. Many authors have studied the problem of measuring inconsistency from pairwise comparison matrices. Saaty (1977) proposed a consistency index defined in terms of the principal eighenvalue, Barzilai (1998) proposed the relative error, and in the literature many other indices of consistency have been proposed, see Chu et al.

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(1979), Cavallo and D'Apuzzo (2009, 2010), Peláez and Lamata (2003), Crawford and Williams (1985), Stein and Mizzi (2007), Shiraishi et al. (1998, 2002), Fedrizzi et al. (1990, 2002, 2007).

In order to take into account some form of inconsistency-based interaction between criteria, the Choquet integral (for reviews see Grabisch and Labreuche (2004, 2008, 2010), Grabisch and Kojadinovich (2008)) is an appropriate aggregation operator. The Choquet integral is defined with respect to a (non additive) capacity and generalizes the weighted arithmetic mean (additive case). In order to control the exponential complexity of the model ($2^n - 2$ real coefficients are required to define a capacity on a set of n elements), Grabisch (1997a) introduced the so called k-additive capacities, see also Grabisch (1997b), and Miranda and Grabisch (1999). The 2-additive case in particular (see Miranda, Grabisch, and Gil, 2005; Mayag, Grabisch, and Labreuche, in press) is a good trade-off between the range of the model and its complexity (only $n(n+1)/2$ real coefficients are required to define a 2-additive capacity).

In this paper we propose an extension of Saaty's AHP based on Choquet integration with respect to a 2-additive capacity: we consider the so-called totally inconsistent matrix induced by Barzilai (1998), and we define a 2-additive capacity on the basis of an appropriate transformation of this matrix. The aggregation scheme is then redefined in terms of the Choquet integration associated to such capacity, thereby extending the usual weighted averaging scheme of Saaty's AHP. A preliminary version of this paper was presented in (Marques Pereira and Bortot, 2004).

An important effect of the new aggregation scheme based on Choquet integration is that of emphasizing (attenuating) the effective priorities of those criteria which have a lower (higher) level of average inconsistency with the remaining ones. This compensatory mechanism that emphasizes some effective priority values and attenuates others is nicely illustrated by the Shapley values associated with the capacity. In our model the Shapley values encode the effective importance weights of the various criteria and, under consistency, the Shapley values coincide with the original priority weights.

2. Extension of Saaty's AHP

Consider a finite set of interacting criteria $N = \{1,2,\ldots,n\}$. A *capacity* is a set function $\mu : 2^N \to [0,1]$ satisfying $\mu(\varnothing) = 0$, $\mu(N) = 1$, and the monotonicity condition: $S \subseteq T \subseteq N \;\Rightarrow\; \mu(S) \le \mu(T)$.

Given a capacity μ, we can define the *Choquet integral* (Choquet, 1953; Grabisch, 1995, 1996) of a vector $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ with respect to μ as

$$\mathrm{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} \tag{1}$$

where the permutation is such that $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. Moreover $A_{(i)} = \{(i),\ldots,(n)\}$ and $A_{(n+1)} = \varnothing$.

Notice that the Choquet integral with respect to an additive capacity μ reduces to a weighted arithmetic mean, whose weights w_i are given by the $\mu(i)$ values. The *importance index* or *Shapley value* (Grabisch and Roubens, 1999) of criterion $i \in N$ with respect to μ is defined as

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus i} \frac{(n-1-t)!(t)!}{n!} \left[\mu(T \cup i) - \mu(T)\right], \qquad \sum_{i=1}^{n} \phi_\mu(i) = 1. \tag{2}$$

It amounts to a weighted average of the marginal contribution of element i with respect to all coalitions $T \subseteq N \setminus i$ and it can be interpreted as an effective importance weight.

Consider now a positive reciprocal $n \times n$ matrix $\mathbf{A}=[a_{ij}]$ with $a_{ij} > 0$ and $a_{ji} = 1/a_{ij}$ for $i, j = 1,\ldots,n$. All pairwise comparison matrices in Saaty's AHP are of this form. However, our model regards only the

single pairwise comparison matrix between criteria at the higher level of the AHP. This is because that matrix is the one that controls the aggregation process: in Saaty's AHP, the aggregation is performed through a weighted average whose weights are the components of the higher level priority vector.
In general, the positive reciprocal matrix $\mathbf{A}$ above is inconsistent, where consistency means $a_{ij} = a_{ik} a_{kj}$ for all $i, j, k = 1, \ldots, n$. However, we can associate to $\mathbf{A}$ a consistent matrix $\tilde{\mathbf{A}}$=[$\tilde{a}_{ij}$] in the following way,

$$\tilde{a}_{ij} = w_i / w_j \qquad w_i = u_i / \sum_{j=1}^{n} u_j \qquad i, j = 1, \ldots, n \tag{3}$$

where u_i is the geometric mean of row i, $u_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}$, and the weights are normalized, $\sum_{i=1}^{n} w_i = 1$.
Given an element a_{ij} of the matrix **A**, we define the *neighborhood* $U(a_{ij})$ as the set of elements of row i and column j of the matrix **A**, that is $U(a_{ij}) = \{a_{ik}, a_{kj} \mid k = 1, \ldots, n\}$. We say that a_{ij} is *locally consistent* if, on average, it is consistent with the elements in its neighborhood,

$$a_{ij} = \tilde{a}_{ij} = \sqrt[n]{\prod_{k=1}^{n} a_{ik} a_{kj}} \qquad i, j = 1, \ldots, n. \tag{4}$$

We now define the *scaling function* $\mathrm{f} : (0, \infty) \to (0,1)$ as shown below,

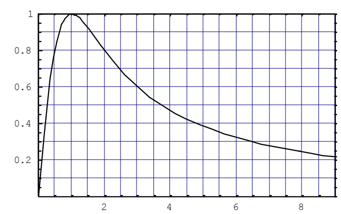

Figure 1. The scaling function $\mathrm{f}(x) = 2/(x + x^{-1})$.

Notice that the scaling function f has a single critical point at $x = 1$, where it reaches the maximum value $\mathrm{f}(1) = 1$. Moreover, the scaling function f has the important property $\mathrm{f}(x) = \mathrm{f}(x^{-1})$ for all $x > 0$.
By means of the scaling function f, we can associate a positive symmetric $n \times n$ matrix **V**=[v_{ij}] to the matrix **A**=[a_{ij}] in the following way,

$$v_{ij} = \mathrm{f}(a_{ij} / \tilde{a}_{ij}) \qquad v_{ij} \in (0,1] \qquad v_{ij} = v_{ji} \qquad i, j = 1, \ldots, n. \tag{5}$$

The fact that the $n \times n$ matrix **V**=[v_{ij}] is symmetric is due to the reciprocity of the positive matrix **A**, plus the fact that $\mathrm{f}(x) = \mathrm{f}(x^{-1})$, for $x > 0$, since $v_{ji} = \mathrm{f}(a_{ji} / \tilde{a}_{ji}) = \mathrm{f}(\tilde{a}_{ij} / a_{ij}) = \mathrm{f}(a_{ij} / \tilde{a}_{ij}) = v_{ij}$.
Notice that $v_{ij} = 1$ if and only if $a_{ij} = \tilde{a}_{ij}$, otherwise $0 < v_{ij} < 1$: the more $a_{ij} / \tilde{a}_{ij}$ differs from 1, the more v_{ij} is close to 0. Therefore we can consider the matrix **V**=[v_{ij}] as a measure of local consistency.
Moreover, note that our matrix **V**=[v_{ij}] can be regarded as a [0,1]-scaled version of the so-called *totally inconsistent matrix* (Barzilai, 1998) associated with the original pairwise comparison matrix **A**=[a_{ij}].
Given a general (typically inconsistent) positive reciprocal matrix **A**=[a_{ij}], one can define a 2-additive capacity $\mu : 2^N \to [0,1]$ in the following way: making use of the Möbius transform m_μ of the capacity μ, we define $m_\mu(i) = 2w_i/(v+1)$ for each singlet $\{i\}$ and $m_\mu(ij) = -2w_i(1 - v_{ij})w_j/(v+1)$ for each doublet $\{i, j\}$, with null higher order terms, where $v_i = \sum_{j=1}^{n} v_{ij} w_j$ and $v = \sum_{i=1}^{n} v_i w_i$ denote weighted averages of local consistency values, with $w_i < v_i \leq 1$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i^2 < v \leq 1$. Then, we define the value of the 2-additive capacity μ on a coalition S as the sum of the singletons and doublets contained in the coalition S, as given by the Möbius transform m_μ,

$$\mu(S) = \sum_{\{i\} \subseteq S} 2w_i/(v+1) + \sum_{\{i,j\} \subseteq S} (-2w_i(1 - v_{ij})w_j)/(v+1) \tag{6}$$

In particular, we have

$$\mu(i) = 2w_i/(v+1) \qquad \mu(ij) = (2w_i + 2w_j - 2w_i(1-v_{ij})\,w_j)/(v+1) \qquad i, j = 1,\ldots,n. \tag{7}$$

The capacity μ satisfies the boundary conditions $\mu(\varnothing) = 0$, $\mu(N) = 1$, and is monotonic and subadditive. The (strict) monotonicity of the capacity is guaranteed by the fact that the positive value w_i associated to each node of the graph dominates (in absolute value) the sum of the negative values $-w_i(1-v_{ij})\,w_j$ associated to the $n-1$ edges connecting that node with the other nodes in the graph,

$$w_i - \sum_{j=1}^{n} w_i(1-v_{ij})\,w_j = w_i - w_i(1-v_i) = w_i v_i > w_i^2 > 0 \qquad i = 1,\ldots,n. \tag{8}$$

This model is an extension of Saaty's AHP: if the matrix $\mathbf{A}$ is consistent, then the capacity μ is additive and the Choquet integral coincides with a weighted arithmetic mean whose weights are as in Saaty's AHP. Using the Möbius transform, one can easily compute the Shapley values ϕ_i, $i = 1,\ldots,n$ associated with the capacity μ defined above,

$$\phi_i = m_\mu(i) + \frac{1}{2}\sum_{j\in N\setminus i} m_\mu(ij) = \frac{2}{v+1}\left(w_i - \frac{1}{2}\sum_{j\in N\setminus i} w_i(1-v_{ij})\,w_j\right) = w_i\,\frac{1+v_i}{v+1} \qquad i = 1,\ldots,n. \tag{9}$$

In our multicriteria aggregation model the Shapley values encode the effective importance weights of the various criteria. When the matrix $\mathbf{A}$ is consistent, we have $v_{ij} = 1$ for all $i, j = 1,\ldots,n$ and the equation above implies that the Shapley values are $\phi_i = w_i$. Otherwise, when $\mathbf{A}$ is inconsistent, we have $\phi_i > w_i$ if $v_i > v$ and $\phi_i < w_i$ if $v_i < v$. In general, the fact that $\mathbf{A}$ is inconsistent changes the original distribution of weights, attenuating the importance values of the more inconsistent criteria (those with higher average inconsistency) and emphasizing the importance values of the more consistent criteria.
In fact, if we compute the second order Taylor expansion of the Shapley values $\phi_i = w_i(1+v_i)/(1+v)$, $i = 1,\ldots,n$, around the consistency condition $v_i \approx 1$, we get

$$\phi_i \approx w_i\left(1 + \frac{1}{4}(v_i - v)(3 - v)\right) \qquad i = 1,\ldots,n. \tag{10}$$

Notice that the second order approximation of the Shapley values is still normalized to unit sum, since $\sum_{i=1}^{n} w_i(v_i - v) = 0$. Moreover, the Taylor expansion shows clearly that, in the small inconsistency approximation, we have $\phi_i > w_i$ if $v_i > v$ and $\phi_i < w_i$ if $v_i < v$, in a compensatory mechanism typical of weighted averaging schemes.

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