ANALYTIC HIERARCHY PROCESS WITH ADJUSTMENTS OF WEIGHTS OF ALTERNATIVES

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ABSTRACT

The Analytic Hierarchy Process (AHP) is decision-making method proposed by T.L.Saaty in the 1970s. The purpose of the paper is to propose a decision making method using AHP with a different purpose, which has questions we use with ANP, and show how to use this by an example. This method will be used when we suppose that there is the table with true estimation values of alternatives with respect to criteria, which are weights of criteria within alternatives at the same time. This method is natural when considering how to judge entrants or players in a competition without absolute estimate values, for example, competitions of singing abilities, musical contests, competitions of ice skating, gymnasia, speech contests and so on. In fact, we will expect the order with weighs of alternatives with respect to criteria like concrete points. If we doubt that such a table is, then we recommend AHP instead. Furthermore, we propose validity test for the result with this method. If the result doesn't pass it, then we recommend giving up looking for such a table and using the result with AHP again.

Keywords: crossed AHP, basis, adjustment, synthesization

1. The purpose of the method with adjustment of weights of alternatives

In this section we clear the purpose of the method I show in this paper. We suppose that a friend broke the leg and is in the hospital. I'd like to select a fruits basket I give him because he likes fruits very much. Then we assume that we select one out of three types of baskets like Table 1 in a shop.

Table 1. The number of fruits in baskets

	Apples	Oranges	Peaches	Bananas
Basket 1	1	4	2	5
Basket 2	2	5	3	1
Basket 3	4	3	5	2

Furthermore, we suppose that we have Table 2 as prior information about his preference about fruits in baskets. Table 2 is including paired comparisons by him.

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Table 2. Preferences among fruits in baskets

	Apples	Oranges	Peaches	Bananas	Priorities
Apples	1	3	4	5	0.542
Oranges	1/3	1	2	4	0.247
Peaches	1/4	1/2	1	2	0.134
Bananas	1/5	1/4	1/2	1	0.077

C.I.=0.0273

Then we have from Tables 1 and 2

$$\begin{pmatrix}\text{Basket 1}\\ \text{Basket 2}\\ \text{Basket 3}\end{pmatrix} = \begin{pmatrix}1 & 4 & 2 & 5\\ 2 & 5 & 3 & 1\\ 4 & 3 & 5 & 2\end{pmatrix}\begin{pmatrix}0.542\\ 0.247\\ 0.134\\ 0.077\end{pmatrix} = \begin{pmatrix}2.183\\ 2.798\\ 3.733\end{pmatrix}$$

and Basket 1 : Basket 2 : Basket 3 = 2.183 : 2.798 : 3.733 = 0.251 : 0.321 : 0.428. The last equal means normalization with AHP. Consequently, we have Basket 1 (0.251) < Basket 2 (0.321) < Basket 3 (0.428). Thus I select Basket 3, which is natural selection.

The purpose of the method is to calculate the last ratios and the order of alternatives with concrete evaluation values like Table 1 or not. This means that we guess the final ratios of weights as if we expect that there exists such a table with true estimation values of alternatives, like Table 1, with respect to criteria. If we doubt that such a table is, for example, according to p.302 in (Saaty, 2006), then we use AHP instead. Furthermore, we have validity test for the result with this method. If the result doesn't absolutely pass it, then we recommend giving up looking for such a table and using the result with AHP again.

2. A validity of this method

We need to see two kinds of validity of our method. Firstly, when we have a table with true estimation values of alternatives with respect to criteria like Table 1, we can easily obtain the ideal result like the final ratio in Section 1. Secondary, when we don't have such a table, we can accept its existence. Certainly, in this case we need validity test for whether it is or not. In this paper we only show first validity. We consider three level hierarchy with a goal *G*, four criteria *A*, *B*, *C* and *D* and three alternatives *a*, *b* and *c*. Then we suppose having Table 3, like Table 1, with true estimation values of alternatives. It's clear to see that if in fact we have Table 3, we easily calculate the final ratio of weights of alternatives.

Table 3. True evaluation values of alternatives with respect to each criterion

	A	B	C	D
a	w_{11}	w_{12}	w_{13}	w_{14}
b	w_{21}	w_{22}	w_{23}	w_{24}
c	w_{31}	w_{32}	w_{33}	w_{34}

We denote weights of criteria or alternatives with respect to Goal or criteria with the same symbols, respectively. For example, weight of criterion *A* is *A* with respect to Goal and weight of alternative *a* is *a* with respect to criterion *B* and so on. Then we assume that $A+B+C+D = 1$. Moreover, we define $V_j=w_{1j}+w_{2j}+w_{3j}$ (j=1,2,3 and 4) and $H_i=w_{i1}+w_{i2}+w_{i3}+w_{i4}$ (i=1,2 and 3). Then we make Table 4 and Table 5 from Table 3.

Table 4 Vertically normalized table of Table 3

	A	B	C	D
a	w_{11}/V_1	w_{12}/V_2	w_{13}/V_3	w_{14}/V_4
b	w_{21}/V_1	w_{22}/V_2	w_{23}/V_3	w_{24}/V_4
c	w_{31}/V_1	w_{32}/V_2	w_{33}/V_3	w_{34}/V_4

Table 5. Horizontally normalized table of Table 3

	A	*B*	*C*	*D*
a	w_{11}/H_1	w_{12}/H_1	w_{13}/H_1	w_{14}/H_1
b	w_{21}/H_2	w_{22}/H_2	w_{23}/H_2	w_{24}/H_2
c	w_{31}/H_3	w_{32}/H_3	w_{33}/H_3	w_{34}/H_3

Then we adjust weights of alternatives in Table 4 with rows in Table 5. For instance, we adjust it with the row of *a* to obtain Table 6. Then the row of *a* in Table 6 is the same as in Table 5 and vertical ratios of weights in Table 6 preserve ones in Table 4. For example, $w_{12}/\mathrm{V}_2 : w_{22}/\mathrm{V}_2 : w_{32}/\mathrm{V}_2 = w_{12}/\mathrm{H}_1 : w_{22}/\mathrm{H}_1 : w_{32}/\mathrm{H}_1$.

Table 6. Table adjusted Table 4 with the row of *a*

	A	*B*	*C*	*D*
a	w_{11}/H_1	w_{12}/H_1	w_{13}/H_1	w_{14}/H_1
b	w_{21}/H_1	w_{22}/H_1	w_{23}/H_1	w_{24}/H_1
c	w_{31}/H_1	w_{32}/H_1	w_{33}/H_1	w_{34}/H_1

Finally, we synthesize Table 6 and weights of criteria, which are *A*, *B*, *C* and *D* and we have

$$\begin{pmatrix}\text{Weight of } a\\ \text{Weight of } b\\ \text{Weight of } c\end{pmatrix} = \begin{pmatrix} w_{11}/\mathrm{H}_1 & w_{12}/\mathrm{H}_1 & w_{13}/\mathrm{H}_1 & w_{14}/\mathrm{H}_1\\ w_{21}/\mathrm{H}_1 & w_{22}/\mathrm{H}_1 & w_{23}/\mathrm{H}_1 & w_{24}/\mathrm{H}_1\\ w_{31}/\mathrm{H}_1 & w_{32}/\mathrm{H}_1 & w_{33}/\mathrm{H}_1 & w_{34}/\mathrm{H}_1\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix} = \frac{1}{\mathrm{H}_1}\begin{pmatrix} Aw_{11}+\mathrm{B}w_{12}+\mathrm{C}w_{13}+\mathrm{D}w_{14}\\ Aw_{21}+\mathrm{B}w_{22}+\mathrm{C}w_{23}+\mathrm{D}w_{24}\\ Aw_{31}+\mathrm{B}w_{32}+\mathrm{C}w_{33}+\mathrm{D}w_{34}\end{pmatrix}.$$

and

$$\begin{aligned}&\text{Weight of } a : \text{Weight of } b : \text{Weight of } c\\ &= \frac{Aw_{11}+Bw_{12}+Cw_{13}+Dw_{14}}{\mathrm{H}_1} : \frac{Aw_{21}+Bw_{22}+Cw_{23}+Dw_{24}}{\mathrm{H}_1} : \frac{Aw_{31}+Bw_{32}+Cw_{33}+Dw_{34}}{\mathrm{H}_1}\\ &= Aw_{11}+Bw_{12}+Cw_{13}+Dw_{14} : Aw_{21}+Bw_{22}+Cw_{23}+Dw_{24} : Aw_{31}+Bw_{32}+Cw_{33}+Dw_{34}\\ &= \frac{Aw_{11}+Bw_{12}+Cw_{13}+Dw_{14}}{\mathrm{T}} : \frac{Aw_{21}+Bw_{22}+Cw_{23}+Dw_{24}}{\mathrm{T}} : \frac{Aw_{31}+Bw_{32}+Cw_{33}+Dw_{34}}{\mathrm{T}},\end{aligned}$$

where $\mathrm{T}=A\mathrm{V}_1+B\mathrm{V}_2+C\mathrm{V}_3+D\mathrm{V}_4$. The final ratio is final answer, which is normalized. Third ratio corresponds to one by directly synthesizing Table 3 and weights of criteria. It is easy to see that we obtain the same final ratio even if we use the row of *b* or *c* instead of *a*, which is called a basis. Consequently, it is shown that when starting from Tables 4 and 5, we have the same as ratio directly calculated from Table 3.

3. Estimation of players in a competition of ice skating

In this section we explain how to use the method by a general example. We show only simple version, though we have two versions. The other is called essential version. Now we consider estimation of players in a competition of figure skating. We remark that marking method here is different from real method as in the Olympics and player's names are not related to it.

Situation: We suppose that we have four criteria, which are Jump, Spin, Balance and Speed, three skaters, whose names are Rochette, Asada and Kim. The order of performance is Rochette, Asada and Kim. Only a judge selects the best skater with our method.

(Step 1) Construct a hierarchy to represent a given problem. The judge constructs Figure 1 as a hierarchy.

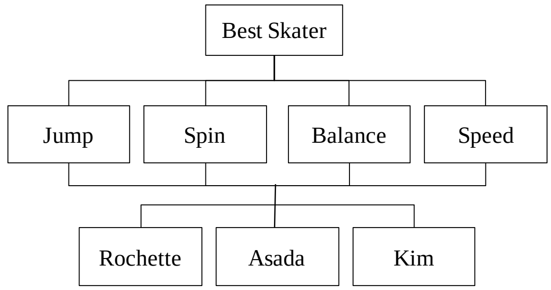

Figure 1. A hierarchy.

Then we suppose that there is Table 7 with true estimation values and calculate the final ratios of weights of alternatives with respect to Goal by it. Certainly, we can't know true values w_{ij}.

Table 7.

	Jump	Spin	Balance	Speed
Rochette	w_{11}	w_{12}	w_{13}	w_{14}
Asada	w_{21}	w_{22}	w_{23}	w_{24}
Kim	w_{31}	w_{32}	w_{33}	w_{34}

(Step 2) Calculate weights of criteria with respect to Goal and weights of alternatives with respect to each criterion. As in the AHP, the judge has Tables 8 and 9 with paired comparison matrices among criteria with respect to Goal and among alternatives with respect to each criterion. Table 9 corresponds to Table 4.

Table 8. Weights of criteria with respect to Goal.

Goal	Weight
Jump	0.657
Spin	0.203
Balance	0.094
Speed	0.046

Table 9. Weights of alternatives with respect to each criterion.

	Jump	Spin	Balance	Speed
Rochette	0.259	0.103	0.540	0.109
Asada	0.105	0.682	0.163	0.309
Kim	0.637	0.216	0.297	0.582

(Step 3) Calculate weights of criteria within alternatives. Here we shall remind the general process in a competition of figure skating. After performance of the first skater Rochette, the judge marks her points with each criterion. For example, her points are 8.5 for Jump, 6.8 for Spin, for 7.2 for Balance and 9.2 for Speed with ten-point full marks. Next, after the second skater Asada performed, the judge marks her points referring to Rochette's points. This process is important and we call it "adjustment of points". Once the judge adjusts points between the first and the second players, he/she doesn't need to adjust points anymore. Certainly, the judge may adjust points among points after each performance of players, but in general it is sufficient to adjust points between first and second. We do adjustment of weights of alternatives in this step.

This step is different from the AHP. We use the following question, for example, on p.167 in (Saaty, 1996) about performance of the first skater Rochette: Which of two criteria is *more characteristic* of Performance of Rochette? and make paired comparison matrices and calculate weights of criteria within alternatives. Summarizing, we have Table 10. This table corresponds to Table 5. We remark that these bases are not units of scale.

Table 10. Weights among criteria within performance of Rochette.

	Bases		
	(Rochette)	(Asada)	(Kim)
Jump	0.565	0.111	0.596
Spin	0.262	0.732	0.266
Balance	0.118	0.049	0.042
Speed	0.055	0.108	0.097

(Step 4) Adjust weights of alternatives. Here we adjust weights of alternatives in Table 9 by Table 10 and make new tables for each basis. Because weights are ratios, we replace weights of Rochette in Table 9 with weight of basis Rochette in Table 10. After this replacement, weights of the other alternatives in Table 9 are changed in order to preserve vertical ratios of weights among alternatives in Table 9 and have Table 11.

Table 11. Adjusted table of Table 9 by weight of basis Rochette in Table 10.

(Rochette)	Jump	Spin	Balance	Speed
Rochette	0.565	0.262	0.118	0.055
Asada	0.229	1.743	0.036	0.156
Kim	1.393	0.552	0.065	0.294

Table 12. Adjusted table of Table 9 by weight of basis Asada in Table 10.

(Asada)	Jump	Spin	Balance	Speed
Rochette	0.273	0.110	0.162	0.038
Asada	0.111	0.732	0.049	0.108
Kim	0.672	0.232	0.089	0.203

Table 13. Adjusted table of Table 9 by weight of basis Kim in Table 10.

(Kim)	Jump	Spin	Balance	Speed
Rochette	0.241	0.126	0.077	0.018
Asada	0.098	0.839	0.023	0.051
Kim	0.596	0.266	0.042	0.097

(Step 5) Synthesize weights of criteria and adjusted weights of alternatives. We synthesize Table 11 and Table 8 as in the AHP:

$$\begin{pmatrix} \text{weight of Rochette} \\ \text{weight of Asada} \\ \text{weight of Kim} \end{pmatrix} = \begin{pmatrix} 0.565 & 0.262 & 0.118 & 0.055 \\ 0.229 & 1.743 & 0.036 & 0.156 \\ 1.393 & 0.552 & 0.065 & 0.294 \end{pmatrix} \begin{pmatrix} 0.657 \\ 0.203 \\ 0.094 \\ 0.046 \end{pmatrix} = \begin{pmatrix} 0.438 \\ 0.512 \\ 1.047 \end{pmatrix}.$$

We have by normalizing Rochette : Asada : Kim = 0.219: 0.257: 0.524. Similarly, we have from Table 12 and Table 8

$$\begin{pmatrix} \text{Rochette} \\ \text{Asada} \\ \text{Kim} \end{pmatrix} = \begin{pmatrix} 0.273 & 0.110 & 0.162 & 0.038 \\ 0.111 & 0.732 & 0.049 & 0.108 \\ 0.672 & 0.232 & 0.089 & 0.203 \end{pmatrix} \begin{pmatrix} 0.657 \\ 0.203 \\ 0.094 \\ 0.046 \end{pmatrix} = \begin{pmatrix} 0.219 \\ 0.231 \\ 0.507 \end{pmatrix}.$$

Then by normalizing we have Rochette : Asada : Kim = 0.229: 0.241: 0.530. Furthermore, we have from Table 13 and Table 8

$$\begin{pmatrix} \text{Rochette} \\ \text{Asada} \\ \text{Kim} \end{pmatrix} = \begin{pmatrix} 0.241 & 0.126 & 0.077 & 0.018 \\ 0.098 & 0.839 & 0.023 & 0.051 \\ 0.596 & 0.266 & 0.042 & 0.097 \end{pmatrix} \begin{pmatrix} 0.657 \\ 0.203 \\ 0.094 \\ 0.046 \end{pmatrix} = \begin{pmatrix} 0.192 \\ 0.239 \\ 0.454 \end{pmatrix}$$

It follows from normalizing that Rochette : Asada : Kim = 0.217: 0.270: 0.513.

(Step 6) Combine weights of alternatives for bases. Finally, we combine the results out of three alternatives by geometric means of row's entries, for example, like the Nash social welfare function.

Table 14. Table with synthesized weight of three alternatives as bases in the final.

	Basis			Geometric	Final
	(Rochette)	(Asada)	(Kim)	Means	Weights
Rochette	0.219	0.229	0.217	0.222	0.222
Asada	0.257	0.241	0.270	0.270	0.256
Kim	0.524	0.530	0.513	0.513	0.522

Consequently, we have Rochette : Asada : Kim =0.222: 0.256: 0.522 and Rochette < Asada < Kim. It follows from this that Kim is selected as the best skater by this judge.

(Validity test) We check the validity of this result. We have the order Rochette < Asada < Kim for all bases. So this result passes this test. If we don't pass this test, then we recommend the following:
(1) If a decision maker chooses just an alternative as a basis, then we don't need this test. This implies that this test always passes if we select just an alternative as a basis.
(2) When Table 10 doesn't pass the test, we may force to do by throwing some bases away out of the set of bases.
(3) When Table 10 doesn't absolutely pass the test, we apply AHP for this problem, while giving up Table 7.

4. Conclusions

In the paper we propose the method like AHP and show how to use it. This method has two types of versions and two cases for each one. We dealt only with simple version. The feature of this method is adjusting weights of alternatives for bases. This method needs questions of ANP, but it have not been cleared the difference from ANP still now.

We may interpret AHP model that if we could select all alternatives at the same time, we would completely satisfy the purpose of Goal while supposing that alternatives or criteria and so on are independent upon each other. Furthermore, when they depend upon each other, AHP model is extended to ANP model. This method corresponds to AHP model and so we may find out a method corresponded to ANP model. We need to research a problem of scale, which is 1 to 9, for this method more precisely.

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