USING ANALYTIC HIERARCHY PROCESS AS A TOOL FOR RANKING OF EFFICIENT UNITS IN DEA MODELS[1]

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ABSTRACT

Traditional data envelopment analysis (DEA) models cannot discriminate among efficient decision making units (DMUs) because all of them have maximum efficiency score 100%. The aim of the paper is to present an original approach for ranking of efficient DMUs based on the analytic hierarchy process (AHP) developed by T. Saaty. The approach runs in two basic steps. The first one is traditional DEA analysis and specification of efficient DMUs. In the second step the AHP model is created with second hierarchical level containing all ratios outputs/inputs. Their priorities are derived as average weights from DEA analysis. Finally the DMUs are evaluated with respect to all criteria and their global priorities are derived. The priorities generate complete ranking of DMUs. The proposed approach is illustrated on a numerical example with real-world background. The results of the DEA/AHP model are compared with other DEA ranking approaches.

Keywords: analytic hierarchy process, data envelopment analysis, efficiency, super-efficiency

1. Introduction

Data envelopment analysis (DEA) is a tool for evaluation and measuring the efficiency of a set of decision making units (DMUs) that consume multiple inputs and produce multiple outputs. Efficiency score which is one of the main information given by DEA models reflects efficiency of transformation of multiple inputs into multiple outputs. In typical case, higher inputs influence the efficiency score in a negative way and in the contrary higher outputs in a positive way. DEA models split the DMUs into two groups – efficient and inefficient. The efficient units are those lying on the efficient frontier which is estimated by the DEA model. Each DMU receives its efficiency score – efficient units 100% and inefficient units (depending on the model used) lower that 100%. The inefficient units can be easily ranked according to their efficiency score. The efficient units cannot be ranked in standard DEA models as their efficiency score is equal to 100%. That is why many models have been formulated in order to allow ranking of efficient units in DEA models. This stream in DEA research is widely developed since 1993 when (Andersen and Petersen, 1993) published their super-efficiency model. This group of models assigns to efficient units the efficiency score higher than 100% which allows their ranking.

Super-efficiency models are based on measuring the distance of the evaluated unit DMU_q from a new efficient frontier given by the removal of this unit from the set of units. Except Andersen and Petersen model several other super-efficiency models have been formulated up to the present. Tone's model (Tone, 2002) is one of the most popular. Other researchers have used different mathematical modelling

[1] The research is supported by the Grant Agency of the Czech Republic - project no. 402/09/0231, and by IGA project F4/18/2011.
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principles for ranking of efficient units. One of the possible principles that can be used for this purpose is the analytic hierarchy process (AHP). The AHP was used for complete ranking of DMUs by (Sinuany-Stern et al., 2000) and (Jablonsky, 2007). An extensive review of ranking models in DEA is given in (Adler et al., 2002).

The aim of the paper is to present an original procedure for ranking of DMUs in DEA models based on combination of AHP and DEA principles. It is organized as follows. The next section contains basic definitions and formulations of standard DEA models and presents several basic super-efficiency models. Section 3 presents an original AHP model for ranking efficient units as mentioned above. Section 4 contains a numerical illustration of the presented model and its comparison with other ranking models. The last section summarizes the results and identifies main directions for future research.

2. Data envelopment analysis models

Let us suppose that the set of DMUs contains n elements. The DMUs are evaluated by m inputs and r outputs with input and output values x_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and y_{kj}, $k = 1, 2, \ldots, r$, $j = 1, 2, \ldots, n$, respectively. The efficiency of the DMU_q can be expressed as the weighted sum of outputs divided by the weighted sum of outputs with weights reflecting the importance of single inputs and outputs v_i, $i = 1, 2, \ldots, m$ and u_k, $k = 1, 2, \ldots, r$ as follows:

$$\theta_q = \frac{\sum_{k=1}^{r} u_k y_{kq}}{\sum_{i=1}^{m} v_i x_{iq}}. \qquad (1)$$

Standard CCR input oriented DEA model formulated by Charnes, Cooper and Rhodes in 1978 consists in maximization of efficiency score (1) of the DMU_q subject to constraints that efficiency scores of all other DMUs are lower or equal than 1. The linearized form of this model is as follows:

$$\begin{aligned}
&\text{minimize} && \theta_q = \sum_{i=1}^{m} v_i x_{iq} && \\
&\text{subject to} && \sum_{k=1}^{r} u_k y_{kq} = 1, && \qquad (2)\\
& && \sum_{k=1}^{r} u_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j = 1,2,\ldots,n,\\
& && u_k, v_i \ge \varepsilon, && k = 1,2,\ldots,r,\ i = 1,2,\ldots,m.
\end{aligned}$$

If the optimal value of the model (2) $\theta^*_q = 1$ then the DMU_q is CCR efficient and it is lying on the CCR efficient frontier. $\theta^*_q > 1$ shows that the DMU_q is not CCR efficient – higher value indicates lower efficiency in this case. This measure is often presented as its reciprocal value, i.e. $1/\theta^*_q$ which is more understandable for decision makers - the higher value is assigned to more efficient units. The model (2) is often referenced as primal CCR output oriented model. Its dual form is sometimes more convenient from computational point of views and its mathematical model is as follows:

$$\begin{aligned}
&\text{maximize} && \theta_q && \\
&\text{subject to} && \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{iq}, && i = 1,2,\ldots,m, \qquad (3)
\end{aligned}$$

$$\sum_{j=1}^{n} y_{kj}\lambda_j - s_k^+ = \theta_q y_{kq}, \qquad k = 1,2,...,r,$$
$$\lambda_j \geq 0, \qquad j = 1,2,\ldots,n,$$

where λ_j, $j = 1, 2, \ldots, n$ are weights of DMUs, s^-_i, $i = 1, 2, \ldots, m$, and s^+_k, $k = 1, 2, \ldots, r$ are slack (surplus) variables and θ_q is the efficiency score of the DMU_q which expresses rate of improvement of outputs in order this unit reaches the efficient frontier. There is a problem that all efficient units identified by the model (2) or (3) have the same efficiency score $\theta_q = 1$. In many cases might be important to have a tool for a diversification and ranking efficient DMUs. That is why many models for classification of efficient units in DEA based on different methodological concepts were formulated by several researchers in the past years. The most important category of such models is represented by super-efficiency DEA models. This class of models supposes removal of the evaluated unit from the set of DMUs and measuring its distance from the new efficient frontier. In super-efficiency models the efficiency scores of inefficient units remain unchanged (lower than 1 for input oriented models and higher than 1 for output oriented) but the efficiency score of efficient units may be higher (lower) than 1. The efficient units can be simply ranked according to their super-efficiency scores. Among super-efficiency models Andersen and Petersen model (AP model) and Tone's SBM model (SBMT model) are the most often used. AP model was formulated in (Andersen and Petersen, 1993). Its output oriented formulation (4) is very close to the standard output oriented formulation of the CCR model (3). Only difference is that the weight of the DMU_q, i.e. λ_q, is set to zero in this model. It causes that the DMU_q is removed from the set of units and the efficient frontier changes its shape after this removal. Super-efficiency score θ_q^{AP} measures the distance of the evaluated DMU_q from the new efficient frontier.

AP model was criticized with respect to its properties many times. That is why several other models were formulated with motivation to improve stability and interpretation of given results. One of them is Tone's super-efficiency model presented in (Tone, 2002) which is a modification of his DEA SBM model. This model removes the evaluated unit DMU_q from the set of units and looks for a DMU* with inputs x_i*, $i = 1, 2, ..., m$, and outputs y_k*, $k = 1, 2, ..., r$, being SBM (and CCR) efficient after this removal. It is clear that all inputs of the unit DMU* have to be greater or equal than inputs of the unit DMU_q and all outputs will be lower or equal comparing to outputs of DMU_q. The super-efficiency measure θ_q^{SBM} is the distance of units DMU_q and DMU* in their input and output space. Mathematical formulation of the SBMT model can be found e.g. in (Tone, 2002) or (Jablonsky, 2007). Tone's model returns optimal objective value greater or equal 1. The optimal efficient score is greater than 1 for efficient DMUs – higher value is assigned to more efficient units. All the SBM inefficient units reach in the super SBM model optimal score 1. That is why this model cannot be used for classification of inefficient units. The model has to be used in two steps. The first step is applied to the entire set of units in order to identify efficient units and classify inefficient units. The second step is the computation of the super-efficiency scores by means of the super SBMT model.

3. Using AHP model for ranking of efficient units

AHP is a powerful tool for analysis of complex decision problems. AHP models organize decision problems as a hierarchical structure with several levels. The first (topmost) level defines the main goal of the decision problem and the last (lowest) level usually describes the decision alternatives (DMUs in our case). The levels in between can contain secondary goals, criteria and sub-criteria of the decision problem. Our aim is to use AHP model for evaluation and discrimination among efficient DMUs. That is why the first level is the goal – evaluation of efficient units, the last level of the hierarchy contains DMUs identified as efficient by appropriate DEA model. The evaluation criteria are inputs and outputs used in DEA analysis but they usually cannot be used directly due to possible high differences among input and/or output values. This problem can be solved by using all possible ratios output/input as decision criteria in AHP model instead of using single inputs and outputs. The ratios can be explained as particular efficiency characteristics. They are easily comparable and that is why AHP can be an ideal tool for evaluation of DMUs using such characteristics. Figure 1 presents a simple hierarchy for evaluation of efficient DMUs in accordance with the presented idea. The second level contains criteria of the evaluation, i.e. particular efficiency characteristics. The last level of the hierarchy contains the DMUs determined as efficient by a DEA model.

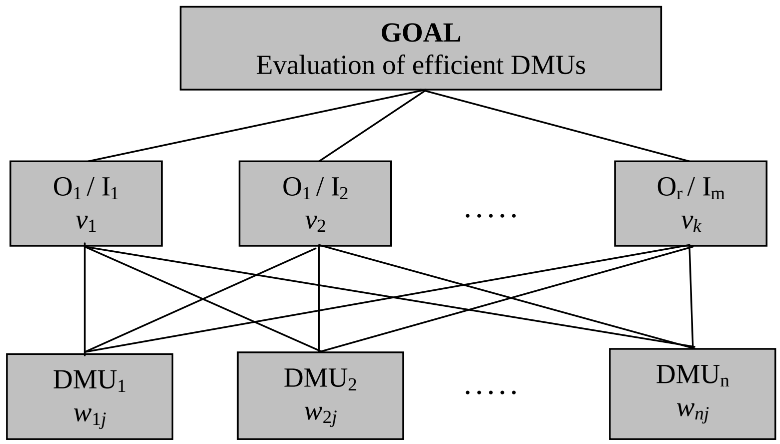

Figure 1 AHP model - evaluation of efficient DMUs.

The weights of the criteria (particular efficiency characteristics) v_k, $k = 1, 2, \ldots, m.r$, are derived by pairwise comparisons in AHP models. This standard approach can be used too but we offer using an alternative way which connects the AHP model with DEA model. We suggest using geometric mean of appropriate input and output weights. They can be taken as:

- M1 - average weights of inputs and outputs of all DMUs given by a DEA model,
- M2 - average weights of inputs and outputs of all efficient DMUs identified by a DEA model,
- M3 – optimal weights of single efficient DMUs.

Using pairwise comparisons of elements on the last level of the hierarchy (efficient DMUs) preference indices of DMUs with respect to the particular efficiency measures w_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m.r$, are derived. Depending on the number of efficient DMUs either AHP with absolute or relative measurement can be used. Global preference indices of all DMUs are given by a simple sum of preference indices w_{ij}:

$$u(DMU_i) = \sum_{j=1}^{m.r} w_{ij}, \; i = 1,2,\ldots,n. \qquad (4)$$

Global preference indices (4) allow final ranking of efficient DMUs.

4. Numerical illustration

The models presented in the last two sections of the paper are illustrated on the set of 194 DMUs – bank branches of one of the Czech commercial banks. The following three inputs and two outputs are used in this study: I_1 – total operational costs in thousands of CZK per year, I_2 – the number of inhabitants within the region of the branch, I_3 – the number of employees, O_1 – value of credits in millions of CZK, and O_2 – the total number of accounts.

By this set of characteristics a business activity of branches is measured. The standard envelopment DEA model with constant returns to scale and output orientation was applied and totally 12 DMUs were identified as efficient. Due to a limited space of the paper the data set is not given here and particular efficiency characteristics for all efficient DMUs are presented in Table 1 only.

DMU	O_1/I_1	O_1/I_2	O_1/I_3	O_2/I_1	O_2/I_2	O_2/I_3
26	0.902	2.006	18.380	0.158	0.352	3227.429
28	0.813	1.743	29.720	0.143	0.307	5242.800
37	0.942	0.424	33.325	0.134	0.060	4739.500
71	0.577	3.561	16.884	0.133	0.822	3896.286
79	0.520	5.722	22.008	0.095	1.046	4021.846
82	0.461	9.472	14.406	0.101	2.085	3170.800
83	0.546	5.715	14.490	0.124	1.295	3282.800
105	0.811	2.324	18.465	0.149	0.427	3390.000
133	0.679	6.514	18.979	0.116	1.113	3241.333
147	0.377	40.369	10.779	0.074	7.893	2107.333
182	0.800	4.624	28.764	0.117	0.674	4189.714
184	0.410	7.353	10.714	0.118	2.121	3091.000

Table 1 Particular efficiency characteristics

The DEA/AHP model can be applied in several steps:

1. Application of a standard DEA model and identification of efficient units (CCR output oriented model in our case).
2. Modification of the original data set for efficient units – criterion values are particular efficiency measures. The efficient DMUs are evaluated by 2.3 = 6 criteria in our example (Table 1).
3. Using AHP model with absolute or relative measurement. The absolute measurement consists in assigning the evaluated DMUs into elements of the evaluation scale. The relative measurement is a standard pairwise comparison approach. In our example we use absolute measurement with five elements evaluation scale: excellent, very good, good, poor and very poor. By pairwise comparisons of evaluation scale's elements their relative strength p_i, i = 1, 2, …, 5 is derived.

	AP model		**Super SBMT**		**DEA/AHP M1**		**DEA/AHP M2**	
DMU	θ_q^{AP}	Rank	θ_q^{SBM}	Rank	*u()*	Rank	*u()*	Rank
26	1.103	6	1.035	9	0.213	4	0.176	7
28	1.227	2	1.100	3	0.243	3	0.219	3
37	1.153	3	1.069	4	0.197	5	0.176	8
71	1.067	9	1.033	10	0.164	10	0.154	11
79	1.093	7	1.040	7	0.173	9	0.190	6
82	1.088	8	1.054	6	0.243	2	0.276	2
83	1.020	11	1.008	11	0.156	12	0.164	10
105	1.005	12	1.003	12	0.178	7	0.147	12
133	1.066	10	1.037	8	0.161	11	0.170	9
147	4.274	1	2.043	1	0.311	1	0.350	1
182	1.134	4	1.112	2	0.177	8	0.193	5
184	1.129	5	1.058	5	0.194	6	0.205	4

Table 2 Ranking of efficient DMUs by different models

4. The DMUs are evaluated according to the criteria (particular efficiency characteristics) by using the elements of the evaluation scale.
5. Using the weights of the criteria global preference indices of DMUs are computed. The following two sets of weights are used in our illustration:

   M1 - average weights of inputs and outputs of all DMUs; the vector of weights in our example is as follows: **v** = (0.1023, 0.2321, 0.0679, 0.1519, 0.3448, 0.1009);
   M2 – average weights of inputs and outputs of efficient DMUs only;
   v = (0.0702, 0.3013, 0.0863, 0.0831, 0.3568, 0.1022).

   The results, i.e. the global preference indices of all efficient DMUs including their ranking, are presented in Table 2 together with results of two mentioned super-efficiency DEA models.

It is clear that both super-efficiency models (AP, SBMT) lead to very close rankings. All models identify the unit 147 as the most efficient. Rankings on other places more or less vary but a more detailed analysis is not subject of this paper. Correlation coefficients of rankings are presented in Table 3.

	SBMT	DEA M1	DEA M2
AP model	0.916	0.671	0.748
SBMT		0.587	0.825
DEA M1			0.741

Table 3 Correlation coefficients of rankings given by different models

5. Conclusions

The aim of the paper was to present the AHP model for ranking of efficient units in DEA models. The results given by this model were compared with other two standard super-efficiency DEA models. The paper does not contain a more detailed analysis of differences in rankings given by presented models. It is an interesting task which can be taken as a starting point for a future research. It can be concentrated on comparison of ranking models with randomly generated data sets of different size and under assumption of different returns to scales.

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