RANK, NORMALIZATION AND IDEALIZATION IN THE ANALYTIC HIERARCHY PROCESS

Thomas L. Saaty
University of Pittsburgh
Saaty@katz.pitt.edu

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Summary: *The AHP uses a fundamental scale of absolute numbers to represent judgments in a paired comparisons matrix. It then derives priorities from the matrix in the form of an absolute scale of relative values. The scale is made relative in two ways: by normalization performed by dividing each value by the sum of all the values; by idealization performed by dividing each value by the largest value among them. Idealization is used when the criteria are independent from the alternatives and also the alternatives are independent from the alternatives. Normalization is used when the criteria depend on the alternatives or when the alternatives depend on the alternatives. The ideas are discussed and illustrated by examples.*

1. Introduction

Cognitive psychologists [Blumenthal, 1977] tell us what everybody inherently knows that people are born with an ability to make comparisons between two alternatives and also to rate alternatives one at a time against an ideal in memory. These are the two ways we have to determine how desirable an outcome of a decision is. To compare an outcome on a criterion relative to other possible outcomes, or in absolute terms by looking at an ideal we know of or can imagine. The first is descriptive and the second is normative. Since memory needs experience to develop ideals, in the life of individuals, comparisons must precede ratings because ideals can only be created through experience. Thus making comparisons is fundamental and intrinsic in us. They are not an intellectual invention nor are they something we can ignore. In addition, to rate alternatives with respect to an ideal, we need to create intensities or levels to indicate how close each alternative comes near the ideal. Even when we use a numerical scale to rate each alternative, in the AHP we must have an idea of how high or how low an alternative falls and in the process subconsciously make comparisons among different levels.

In rating alternatives, one evaluates each alternative in turn by assigning a value for each criterion. In that case the alternatives have to be considered as unconditionally independent of one another. The presence or absence of other alternatives whether they are relevant or irrelevant to that decision, has no effect on the ranking of any one of them. We call this kind of ranking of alternatives with respect to an ideal (an arbitrarily chosen fixed reference point) *absolute measurement.*

Unlike rating alternatives with respect to the best possible ideal alternative, comparing alternatives requires that we directly or indirectly compare each alternative with every other alternative. In that case an alternative that is ideally poor on an attribute could have a relatively high priority when compared with still poorer alternatives on that attribute but have low priority on another attribute where it is ideally good but is compared with better-valued alternatives. Thus the final rank of any alternative depends on the quality of the alternatives with which it is compared. Here by making comparisons among alternatives, the priority of any alternative may be influenced not only by how many alternatives it is compared with but also by how many copies of it there are. Thus rather than being unconditionally independent of each other the alternatives are in fact conditionally independent. We call this kind of ranking of alternatives with respect to other alternatives *relative measurement.*

In the AHP paired comparison judgments are entered in a reciprocal matrix. Absolute numbers from a fundamental scale are used to represent these judgments. From the matrix an absolute scale of relative values is derived and normalized (dividing each value by the sum of all the values) that is used when conditional dependence is needed, or by idealization (dividing each value by the largest value of any alternative) used when conditional dependence is not needed. Both these modes the first called distributive and the second ideal are required for use in relative measurement. In the consistent case the priorities are obtained by adding the entries in any column that are absolute number and divided by the total again an absolute number. In the inconsistent case one solves a system of linear equations that have absolute scale coefficients again obtaining an absolute scale solution that becomes relative on normalization or idealization. Note that the fundamental scale may be thought of as the ratio of absolute numbers like the number of people in different rooms or even of measurement from a ratio scale but because the ratio is taken that ratio and also each of its components may as well be regarded as belonging to some kind of an absolute scale of numbers because it is always used in ratio form and not by itself. Thus the AHP uses only absolute scale numbers for judgments and for priorities.

The dogma of rank preservation asserts that when independent alternatives are rated or measured one at a time on a numerical scale with respect to an ideal, adding or deleting an alternative should have no effect on their rank unless judgments are changed or new criteria are introduced. No one would contest this dogma if the only way to rank alternatives were one at a time with respect to an ideal. It is trivial because in that case they are independent and no alternative can interfere with the ranking of any other alternative. It is questionable and is supported by numerous counterexamples when indiscriminately applied to relative measurement.

We intend to show that in practice and depending on the kind of decision one has to make, both normalization and idealization are necessary in order to allow rank to change or prevent it from changing, respectively.

2. Normalization (any dependence on or among the alternatives) and Idealization (total independence from or among the alternatives)

a) Need for normalization when an existing unit of measurement is used for all the criteria

When there is a single unit of measurement for all the criteria, normalization is important for converting the measurements of alternatives to relative values and synthesizing in order to obtain the right answer. Let us see first what happens when we go from scale measurements to relative values with respect to two criteria by using the same kind of measurement such as dollars for two criteria and give the measurements of three alternatives for each. We then add them and then normalize them by dividing by their total with respect to both criteria as in Table 1 to obtain their relative overall outcome.

Table 1. Scale Measurement Converted to Relative Measurement

Alternatives	Criterion C_1	Criterion C_2	Sums	Relative Value of Sums
A_1	1	3	4	4/18 = .222
A_2	2	4	6	6/18 = .333
A_3	3	5	8	8/18 = .444

Normalization is Basic in Relative Measurement

To obtain the relative values in the last column of this table, given that the numbers in the two columns under the criteria are represented in form relative to each other, the AHP requires that the criteria be assigned priorities in the following way. One adds the measurement values under each and divides it by the sum of the measurements with respect to all the other criteria measured on the same scale. This gives the priority of that criterion for that unit of measurement. Multiplying the relative values of the alternatives by the relative values of the criteria, and adding gives the final column of Table 2. Each of the middle three columns of Table 2 gives the value and the value normalized (relative value) in that column.

Table 2. Scale Measurement Converted to Relative Measurement

Alternatives	Criterion C_1 Normalized weight = 6/18	Criterion C_2 Normalized weight = 12/18	Sums and Normalized Sums	AHP Synthesized Weighted Relative Values
A_1	1 1/6	3 3/12	4 4/18	4/18 = .222
A_2	2 2/6	4 4/12	6 6/18	6/18 = .333
A_3	3 3/6	5 5/12	8 8/18	8/18 = .444

The outcome in the last column coincides with the last column of Table 1, as it should. More generally, normalization is always needed when the criteria depend on the alternatives as in the Analytic network Process (ANP).

One thing we learn from this example is that if we add new alternatives, the ratios of the priorities of the old alternatives remain the same. Let us prove it for example in the case of two criteria C_1 and C_2. We begin with two alternatives A and B, whose priorities under C_1 and C_2 are respectively, a_i and b_i $i = 1,2$ which in relative form are

$$a_i / \sum_{i=1}^{2} a_i \text{ and } \mathrm{b}_i / \sum_{i=1}^{2} b_i .$$

The weights of C_1 and C_2 are respectively

$$\sum_{i=1}^{2} a_i / (\sum_{i=1}^{2} a_i + \sum_{i=1}^{2} b_i), \sum_{i=1}^{2} b_i / (\sum_{i=1}^{2} a_i + \sum_{i=1}^{2} b_i) .$$

Synthesizing by weighting and adding yields for the overall priorities of A and B respectively

$$(a_1 + b_1) / (\sum_{i=1}^{2} a_i + \sum_{i=1}^{2} b_i), \text{ and } (a_2 + b_2) / (\sum_{i=1}^{2} a_i + \sum_{i=1}^{2} b_i) .$$

The ratio of these priorities is $(a_1 + b_1)/(a_2 + b_2)$ which only depends on their values and not on the priorities of the criteria. We note that the sum of the values of the alternatives is used to normalize the value of each alternative by dividing by it. But this value is also the numerator of the priority of that criterion and cancels out in the weighting process leaving the sum of the values of the alternatives under both criteria in the denominator of the final result. This sum in turn cancels in taking the ratio of the priorities of *A* and B. Now it is clear that if we add a third alternative *C*, this ratio of the priorities of *A* and B remains unaffected by the change in the priorities of the criteria due to *C*. We conclude that in this case where the priorities of the criteria depend on the alternatives, the ratio of the priorities of the alternatives is invariant to adding a new alternative. This invariance should also hold in the stronger case when the criteria are independent of the alternatives. Thus when the ideal mode is used the ideal must be preserved once and for all so that when new alternatives are added, they are compared with the old ideal and thus the ratios among the existing alternatives can be preserved.

b) Normalization is needed when the criteria are independent of the alternatives where the quality and number of all the alternatives are important in the decision

Normalization is particularly needed in the case of allocating a limited resource in order to determine the relative values of the outcomes.

Consider a manufacturing plant consisting of three divisions whose total capacity is limited to 100,000 tons. Suppose that amounts of products A, B, C are manufactured in the proportions shown in Table 3. Assume that each division can produce 1/3 of the total, i.e., 33,333.33 tons. The amounts produced are shown in Table 4. and the productions are B>A>C. On adding D and reducing production to proportions and amounts as in Tables 5 and 6 we have the following order of production A>B=D>C, having preserved the previous proportions of production between the divisions. By preserving order as we indicated previously to be essential, there is reversal in the order of production between A and B after adding D.

Table 3: Proportions of Production of 3 Products

	Divisions		
Products	I	II	III
A	.0909	.8182	.4444
B	.8182	.0909	.5000
C	.0909	.0909	.0556

Table 4: Amounts of Production of 3 Products

	Divisions			
Products	I	II	III	Total
A	3030	27,273.33	14,813.33	45,116.66
B	27,273.33	3030	16,666.67	46,970.00
C	3030	3030	1,851.85	7,911.85

Table 5: Proportions of Production of 4 Products

	Divisions		
Products	I	II	III
A	.05	.75	.2964
B	.45	.0833	.3333
C	.05	.0833	.0370
D	.45	.0833	.3333

Table 6: Amounts of Production of 4 Products

	Divisions			
Products	I	II	III	Total
A	1,666.67	25,000	9,880.00	36,546.67
B	15,000.00	2,776.67	11,111.11	28,887.78
C	1,666.67	2,776.67	1,234.57	5,677.91
D	15,000.00	2,776.67	11,111.11	28,887.78

c) Need for idealization to preserve rank when new alternatives are added and the priorities of the alternatives do not depend on the number of alternatives and on their quality

We begin with two alternatives A and B. We have on pairwise comparing them in Table 7 with respect to the criteria Efficiency and Cost whose priorities are .5 each. :

Table 7 An Example of Rank Reversal with Change in Ideal

	Efficiency (.5)					Cost (.5)					Composite Ideal	
	0.5	0.5				0.5	0.5					
	A	B	Norm	Ideal		A	B	Norm	Ideal	Comp Dist	Comp	Renorm.
A	1	3	0.75	1	A	1	0.5	0.33	0.5	0.542	0.75	0.5294
B	0.333	1	0.25	0.333	B	2	1	0.67	1	0.458	0.6667	0.4706
										1	1.4167	1

The question now is whether to normalize by dividing the weights of the alternatives by their sum (distributive mode) or idealize by dividing by the weight of the largest alternative (ideal mode). The distributive mode gives A = .54 and B = .46 while the normalized ideal mode gives A = .53 and B = .47. Now, if we add C that is a relevant alternative under efficiency, because it dominates both A and B we obtain as in Table 8:

Table 8 An Example with Rank Reversal with Change in Ideal

	Efficiency (.5)						Cost (.5)						Composite Ideal	
	A	B	C	Norm	Ideal		A	B	C	Norm	Ideal	Comp Dist	Comp	Renorm.
A	1	3	0.5	0.3	0.5	A	1	0.5	4	0.308	0.5	0.3038	0.500	0.304
B	0.333	1	0.167	0.1	0.17	B	2	1	8	0.615	1	0.3577	0.583	0.354
C	2	6	1	0.6	1	C	0.25	0.13	1	0.077	0.125	0.3385	0.563	0.342
							3.25	1.63	13			1	1.6458	1

Now, the distributive mode gives A= .30, B= .36 and C= .34 with rank reversal between A and B, and the normalized ideal mode gives A= .30, B= .35 and C= .34 again with rank reversal. There is rank reversal with both the distributive and ideal modes because C is dominant with respect to efficiency. Now the old ranks of *A* and *B* can be preserved if we maintain the original ideals under each criterion and for each criterion we compare the new alternatives only with the ideal, allowing its value to go above its value of one if necessary. One can even compare it with several of the old alternatives, preserving their relative values but improving any inconsistency only with respect to these values and in view of that adopting a final scale value for the new alternative. In that case we have for the above example the following (Table 9):

Table 9 Preserving Rank with no Change in Ideal

	Efficiency (.5)						Cost (.5)						Composite Ideal	
	A	B	C	Norm	Old Ideal		A	B	C	Norm	Ideal	Comp Dist	Comp	Renorm.
A	1	3	0.5	0.3	1	A	1	0.5	4	0.308	0.5	0.3038	0.7500	0.3024
B	0.333	1	0.167	0.1	0.333	B	2	1	8	0.615	1	0.3577	0.6667	0.2689
C	2	6	1	0.6	2	C	0.25	0.125	1	0.077	0.125	0.3385	1.0625	0.4285
							3.25	1.63	13			1	2.4792	1

and we have no rank reversal. In this case we have idealized only once by using the initial set of alternatives but never after so that rank would be preserved from then on.

3. The Non-Criteria of Manyness and Uniqueness: Copies and the Number of Alternatives

How many alternatives there are is sometimes a criterion that influences the rank of an alternative. However, number cannot be included as a factor in a decision structure because first number is not a property of any alternative, and second if number were to be used as a criterion an alternative becomes dependent on how many other alternatives there are to determine if it is unique. That would imply that the alternatives are not independent and contradict the assumption of independence in which case anything can happen to the rank of the alternatives. In such cases the alternatives are evaluated in the context of the supermatrix where both independence and dependence are possible, and give back the correct priorities without need to use “number” as a criterion. It seems paradoxical that we have to explain the effect of number without including number as a criterion. In relative measurement we not only need to know the relative values of things but also their actual values. Normalization is a way of making that possible.

Consider the example of the lady who shopped for hats and found two hats she liked almost equally only to discover that there were many copies of the one she liked better, and she bought the other. One would say she did not want to be seen wearing a hat that is worn by many other women, but she only became conscious of that because she learned that there were many hats of the same kind. Now assume that instead of the hats it was computers. In that case she would not change her mind to buy the best computer. The judgments are identical in both cases yet the decision is different. What criterion can one use to account for the difference without violating independence? To say that the hats and computers are independently evaluated among themselves prevents one from recognizing that there are many others, yet number has an effect and any criterion that takes it into consideration makes the alternatives dependent because of number. Changing ones preference because of knowledge that there are many of the same alternative, assumes there is dependence. It appears that whether number should or should not influence the outcome is up to the decision maker, and should not be legislated once and for all because it can go either way, number influences one’s decision or it does not.

Note that all these questions that arise in real life also arise in comparisons and relative measurement. They do not occur in rating because the alternatives are assumed to be unconditionally independent and cannot be treated by absolute measurement or by utility theory that rate alternatives one at a time. Such concerns are prevalent in real life and should not be ignored by forcing one to only do rating instead of making comparisons. The problems of copies and non-uniqueness, phantoms, decoys and other paradoxical cases that cannot be accommodated through absolute measurement need relative measurement to account for why they happen. If there is some other “magical” that we have to understand what happens in the real world, psychologists do not seem to be aware of it to tell us about it and about how we have been using it in an unconscious way. Absolute measurement is a normative and a forced way for us to make it convenient to deal with the world. It is inadequate to deal with all the complexity around us because of the effects of the environment that we don’t know well enough.

4. The Need for Weighting and Adding in Multicriteria Decision Making

People have proposed that when different tangible criteria which may be considered as intangible among themselves to synthesize the weights of the alternatives one can use other than additive weighting, such as the method described in the next section. Let us show that such an approach would lead to bad answers. We note that the AHP is based on the fact that to have a priority order that can be aggregated into a single priority order, there has to be proportionality among the different measurements of the alternatives, and that this proportionality arises out of the order that is in our minds. Without proportion one can never be certain about one’s ability to order alternatives under different criteria and combine the results into a single overall answer. Proportionality among the criteria means that they must have priorities to be traded off and used to weight and combine the alternatives measured under them particularly when the alternatives are homogeneous. Other methods violate this basic homogeneity assumption that is needed in the paired comparisons process.

5. Conclusions

Both the distributive and ideal modes are necessary for use in the AHP. We have shown that idealization is essential and is independent of what number crunching method one may use. There are scholars who have made it an obsession to find ways to avoid rank reversal in every decision and wish to alter the synthesis of the AHP away from normalization or idealization. They are likely to obtain outcomes that are not compatible with what the real outcome of a decision should be, because in decision-making we also want uniqueness of the priority of the decision we make. We conclude by offering the reader such an unproductive attempt by one group.

The multiplicative approach to the AHP uses the familiar methods of taking the geometric mean to obtain the priorities of the alternatives for each criterion without normalization, and then raising them to the powers of the criteria and again taking the geometric mean to perform synthesis in a distorted way to always preserve rank. It is unworkable for at least two reasons discussed below.

First let us see what it yields for our simple example of relative values with two criteria measured in dollars given in Table 1. We have Table 10.

Table 10 Multiplicative Synthesis

Alternatives	Criterion C_1 Normalized weight = 6/18		Criterion C_2 Normalized weight = 12/18		Multiplicative Synthesis without Normalization	Multiplicative Synthesis Outcome Normalized
A_1	1	1/6	3	3/12	$[(1)^{1/3}.(1)^{2/3}]^{1/2} = 1$	$.207 \neq .222$
A_2	2	2/6	4	4/12	$[(2)^{1/3}.(4)^{2/3}]^{1/2} = 1.781$	$.369 \neq .333$
A_3	3	3/6	5	5/12	$[(3)^{1/3}.(5)^{2/3}]^{1/2} = 2.054$	$.425 \neq .444$

As can be seen, the values in the last column are different from those of the last column of Table 1.

Second and more seriously, the multiplicative method has an untenable mathematical problem. Assume that an alternative has a priority .2 with respect to each of two criteria whose respective priorities are.3 and .5. It is logical to assume that this alternative should have a higher priority with respect to the more important criterion, the one with the value of .5, after the weighting is performed. But $.2^{.5} < .2^{.3}$ and alas it does not, it has a smaller priority. One would think that the procedure of ranking in this way would have been abandoned at first knowledge of this observation. Its advocates say no decision theory is perfect. What would happen to mathematics if such an excuse were given to justify all its wrong ideas of which there are many more than there are correct ones? Keep them all in because no mathematics is perfect? Would the reader recommend using the multiplicative method knowing this counter intuitive behavior? There have been loud cries made against it at institutes of learning in parts of the world where people who had been using “conventional” AHP are now being told they are required by the terms of the contract with one international organization to use the multiplicative method if funding is to be expected.

References

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