PARAMETERS OF OPTIMUM HIERARCHY STRUCTURE IN AHP

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ABSTRACT

A problem of finding optimal hierarchy parameters for a given number of alternatives is considered. Hierarchical structures are widely used in the Analytic Hierarchy Process, Conjoint Analysis, and various other methods of Multiple Criteria Decision Making. The suggested approach is based on minimizing the objective of total pair comparisons across all the hierarchy structure. For an optimal hierarchy, the minimum effort is needed for eliciting data and synthesizing the local preferences across the hierarchy to get the global priorities or utilities. The obtained analytical and numerical results show how to choose the optimal structuring of the alternatives into groups by sub-criteria, criteria, and hyper-criteria by many-level hierarchy. The obtained results are beneficial for practical managerial decision making in the complex problems with numerous alternatives.

Keywords: AHP, Hierarchy Optimization, Pair Comparisons.

1. Introduction

Designing a hierarchy usually presents the first task in the AHP or other approaches to multi-attribute decision making. A hierarchy configuration can be outlined by scrutiny of the connections among the alternatives for combining them into groups of different criteria. Then all pair comparisons data can be elicited from an expert for each level of the hierarchy, thus, among the items by each criterion, and among the criteria themselves. Statistical design for reducing the number of pair comparisons is applied in the case of many respondents, especially, in the conjoint analysis. The paper considers a problem of assembling a hierarchy structure in the optimizing approach of finding the minimum number of needed pair comparisons among the alternatives, sub-criteria, and criteria. In many situations a researcher does not have prior information on a possible hierarchical design, so using the results of the optimization technique can help to facilitate the data eliciting process for local and global priorities. More specifically, for a given number n of all the alternatives, it is possible to estimate how many of them should be combined into each group of the lower level, and how many these groups of the criteria are needed at the upper levels so that the total number of all the required pair comparisons across the hierarchy structure reaches its minimum. This approach can serve to various practical aims of managerial decision makers.

2. Literature Review

Researchers working with AHP methodology and applications should be familiar with Saaty's classical monographs (1980, 1996), and others. But what is less known, T. Saaty is the author of the monograph (1970) on the optimization in integers, for instance, developed and widely applied in linear and goal programming and various other operation research algorithms. The current work is motivated by the ideas in Saaty (1970), although uses not the complex techniques of integer optimization but only a general setup and approximate solutions of finding integer solutions by the rounded continuous optimal values of the hierarchical structures. Applications of these results have been used in solving practical complex problems in marketing research (Lipovetsky, 2006, 2009).

3. Hypotheses/Objectives

The main aim of this study consists in an advance evaluating of a set of appropriate parameters of a hierarchy sizes which then can be used in the time-money-efforts-saving AHP data eliciting process because it would need finding the minimum number of all the pair comparisons across all the hierarchy.

4. Research Design/Methodology

Let us consider a simple case of two-level hierarchy, with q criteria compared at the upper level, and $m_1, m_2, \ldots, m_q$ alternatives compared within the 1st, 2nd, etc., till the q-th criterion, respectively. Total of all the alternatives is a given constant n:

$$n = m_1 + m_2 + \ldots + m_q . \tag{1}$$

The total of all paired combinations among the criteria and alternatives by the criteria equals:

$$T = \frac{q(q-1)}{2} + \sum_{j=1}^{q} \frac{m_j(m_j - 1)}{2} . \tag{2}$$

Adding the restriction (1) yields the conditional objective:

$$T = \frac{q(q-1)}{2} + \sum_{j=1}^{q} \frac{m_j(m_j - 1)}{2} - \lambda \left(\sum_{j=1}^{q} m_j - n \right), \tag{3}$$

where λ is a Lagrange term. The quadratic items in (3) describe the convex functions, so the minimum by m_j can be defined by the condition of each partial derivative equals zero:

$$\frac{\partial T}{\partial m_j} = m_j - \frac{1}{2} - \lambda = 0 , \qquad j = 1, 2, \ldots , q . \tag{4}$$

Summing equations (5) by j and dividing by q yields:

$$\lambda + \frac{1}{2} = \frac{1}{q} \sum_{j=1}^{q} m_j = \frac{n}{q} \equiv m , \tag{5}$$

where m denotes a mean value of the sizes of all groups of alternatives (1). Substituting (5) into equations (4) shows that

$$m_j = m = n/q, \qquad j = 1, 2, \ldots , q , \tag{6}$$

so the minimum number of the pairs (2) can be reached with equal division of all n alternatives by m of them into each of q criteria groups. Using (6) in (2) yields the total as follows:

$$T = \frac{q(q-1)}{2} + q \frac{(n/q)(n/q-1)}{2} = \frac{q(q-1)}{2} + \frac{n}{2}\left(\frac{n}{q} - 1 \right), \tag{7}$$

where the first item in sum corresponds to the number of combinations at the upper level of q criteria, and the second item is the number of combinations in q groups of m alternatives in each at the lower level of the hierarchy.
The formula (7) expresses the total number of pair comparisons via a given constant n of all alternatives and the unknown number q of criteria. Minimizing the objective (7) yields:

$$\frac{dT}{dq} = q - \frac{1}{2} - \frac{n^2}{2q^2} = 0\,, \qquad \frac{d^2T}{dq^2} = 1 + \frac{n^2}{q^3} > 0\,. \tag{8}$$

So the first order derivative corresponds to minimum and can be reduced to the equation:

$$q^3 - q^2/2 - n^2/2 = 0\,. \tag{9}$$

This cubic equation has only one real root which can be simplified to the expression:

$$q = 2^{-1/3} n^{2/3} = 0.79\, n^{2/3}\,. \tag{10}$$

Then for the optimum number (10) of criteria, the number of alternatives within each criterion can be estimated by (6) as:

$$m = n/q = 2^{1/3} n^{1/3} = 1.26\, n^{1/3}\,. \tag{11}$$

Then the minimum number of total pair comparisons (7) can be reduced to the formula:

$$T_{\min} = \frac{3}{2 \cdot 2^{2/3}} n^{4/3} = 0.94\, n^{4/3}\,, \tag{12}$$

so the rate $n^{4/3}$ of the pairs' number increase as n grows is slightly higher than a linear one.
The total number of all pairs without reducing the number of needed comparisons can be estimated as number of combinations (1) from n alternatives, $T_{\max} = n(n-1)/2$, which is the quadratic function of n. The quotient of the minimum (12) to this maximum of the needed number of pairs can be presented as:

$$\frac{T_{\min}}{T_{\max}} \approx \frac{0.94\, n^{4/3}}{n(n-1)/2} \approx 1.88\, n^{-2/3}\,. \tag{13}$$

With n increasing this ratio is quickly diminishing, so the structuring of the alternatives into two hierarchy levels significantly reduces the needed work of data eliciting and processing.

5. Data/Model Analysis

For a given number of the alternatives n from 5 to 100, the results of the hierarchy parameters' evaluation by the formulae (10)-(13) are presented in the table given in Appendix. Of course, we have to pick the values q and m as integers in a vicinity of the values shown in that table. For instance, with n=53 alternatives, the probable best value for the number of criteria should be about q=11, by m=5 items compared within each criterion. Then the total number of all pair comparisons can be reduced from the maximum value Tmax=1378 to Tmin=156, so almost by 9 times. More complicated hierarchies with three and more levels of criteria can be studied in a similar approach.

6. Limitations

In the described approach we can only obtain very rough estimates of the optimum hierarchy structures. More exact values can be found by the methods of integer quadratic programming and other techniques of integer optimization (Saaty, 1970).

7. Conclusions

A problem of finding an optimal structure of the hierarchy for a given number of alternatives is considered. The suggested technique is based on minimizing the objective of total pair comparisons across all the hierarchy levels. The obtained analytical and numerical results show how to choose the optimal structuring of the alternatives into groups by criteria. Such an approach can be extended to a many level hierarchy as well. This approach can significantly reduce the efforts in eliciting data and synthesizing the local preferences into global priorities, and be beneficial for practical managerial decision making in complex problems with numerous alternatives. Future developments should be based on more exact methods of optimization in integers (see Saaty, 1970).

8. Key References

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9. Appendix. **Two-Level Hierarchy Optimal Parameters.**

n	*m*	*q*	*Tmin*	*Tmax*		*n*	*m*	*q*	*Tmin*	*Tmax*
5	2.2	2.3	4.4	10		53	4.7	11.2	156.0	1378
6	2.3	2.6	6.0	15		54	4.8	11.3	160.2	1431
7	2.4	2.9	7.7	21		55	4.8	11.5	164.4	1485
8	2.5	3.2	9.5	28		56	4.8	11.6	168.6	1540
9	2.6	3.4	11.5	36		57	4.8	11.8	172.9	1596
10	2.7	3.7	13.5	45		58	4.9	11.9	177.2	1653
11	2.8	3.9	15.7	55		59	4.9	12.0	181.5	1711
12	2.9	4.2	17.9	66		60	4.9	12.2	185.9	1770
13	3.0	4.4	20.2	78		61	5.0	12.3	190.3	1830
14	3.0	4.6	22.6	91		62	5.0	12.4	194.7	1891
15	3.1	4.8	25.0	105		63	5.0	12.6	199.1	1953
16	3.2	5.0	27.6	120		64	5.0	12.7	203.6	2016
17	3.2	5.2	30.2	136		65	5.1	12.8	208.0	2080
18	3.3	5.5	32.9	153		66	5.1	13.0	212.6	2145
19	3.4	5.7	35.6	171		67	5.1	13.1	217.1	2211

20	3.4	5.8	38.4	190	68	5.1	13.2	221.7	2278
21	3.5	6.0	41.2	210	69	5.2	13.4	226.2	2346
22	3.5	6.2	44.1	231	70	5.2	13.5	230.9	2415
23	3.6	6.4	47.1	253	71	5.2	13.6	235.5	2485
24	3.6	6.6	50.1	276	72	5.2	13.7	240.2	2556
25	3.7	6.8	53.2	300	73	5.3	13.9	244.9	2628
26	3.7	7.0	56.3	325	74	5.3	14.0	249.6	2701
27	3.8	7.1	59.5	351	75	5.3	14.1	254.3	2775
28	3.8	7.3	62.7	378	76	5.3	14.2	259.1	2850
29	3.9	7.5	65.9	406	77	5.4	14.4	263.9	2926
30	3.9	7.7	69.3	435	78	5.4	14.5	268.7	3003
31	4.0	7.8	72.6	465	79	5.4	14.6	273.5	3081
32	4.0	8.0	76.0	496	80	5.4	14.7	278.4	3160
33	4.0	8.2	79.4	528	81	5.5	14.9	283.2	3240
34	4.1	8.3	82.9	561	82	5.5	15.0	288.1	3321
35	4.1	8.5	86.4	595	83	5.5	15.1	293.1	3403
36	4.2	8.7	90.0	630	84	5.5	15.2	298.0	3486
37	4.2	8.8	93.6	666	85	5.5	15.3	303.0	3570
38	4.2	9.0	97.2	703	86	5.6	15.5	308.0	3655
39	4.3	9.1	100.9	741	87	5.6	15.6	313.0	3741
40	4.3	9.3	104.6	780	88	5.6	15.7	318.0	3828
41	4.3	9.4	108.4	820	89	5.6	15.8	323.1	3916
42	4.4	9.6	112.2	861	90	5.6	15.9	328.1	4005
43	4.4	9.7	116.0	903	91	5.7	16.1	333.2	4095
44	4.4	9.9	119.8	946	92	5.7	16.2	338.4	4186
45	4.5	10.0	123.7	990	93	5.7	16.3	343.5	4278
46	4.5	10.2	127.7	1035	94	5.7	16.4	348.7	4371
47	4.5	10.3	131.6	1081	95	5.7	16.5	353.8	4465
48	4.6	10.5	135.6	1128	96	5.8	16.6	359.0	4560
49	4.6	10.6	139.6	1176	97	5.8	16.8	364.3	4656
50	4.6	10.8	143.7	1225	98	5.8	16.9	369.5	4753
51	4.7	10.9	147.8	1275	99	5.8	17.0	374.8	4851
52	4.7	11.1	151.9	1326	100	5.8	17.1	380.1	4950