Resolving Rank Reversal in Consistent and Independent AHP Models

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ABSTRACT

An open question that has existed for some time now is how to preserve rank in the AHP when a new alternative is added or when one is deleted. The essential conditions are that all judgments be consistent and all elements are independent; these have not been fully considered by the AHP critics and defenders. When a new alternative is added or when one is deleted, rank should be preserved when the conditions are satisfied. The weighted geometric mean aggregation rule is proposed to achieve the desired outcome. A proof demonstrates that the weighted geometric mean aggregation rule can preserve rank in the normalized priority vector. Finally, the causes of rank reversal are analyzed: the principal eigenvector approach and the relative mode, and derive that they are not the real reasons of rank reversal.
Keywords: decision analysis, AHP, rank reversal, aggregation rule

1. Introduction

We propose a method is through the weighted geometric mean aggregation rule to achieve rank preservation. However, the usage of the modified geometric mean aggregation rule must also address two other concerns: 1) the local priorities should be obtained through the principal eigenvector approach and one should use the modified geometric mean aggregation to synthesize the criteria clusters in the overall model; 2) the shortcomings of synthesizing with the geometric mean should be overcome in the aggregation process (detailed in section 3.3). We not only prove that the modified geometric mean aggregation rule can force rank preservation through the principal eigenvector approach, but also overcome the shortcomings of the geometric mean in the aggregation process. Finally, two traditional numerical examples of rank reversal from the literature are presented to check the validity of the proposed method.

2. Preliminaries

The rank reversal phenomenon can be described as when three alternatives (*A*, *B*, and *C*) are ranked in order $B > A > C$ by the AHP. Then when another alternative *D*, which is an exact copy of *B*, is added, the alternatives are ranked in the order $A > B = D > C$; thus, the introduction of an irrelevant alternative causes *A* and *B* to switch order. The following is the example provided by Belton and Gear .

3. The reason of rank reversal and its resolution

3.1 The reasons of rank reversal

Saaty explained that the major objection raised against the AHP by practitioners of utility theory has been the issue of rank reversal. In reviewing the critiques of rank reversal in the AHP, three reasons can be identified.

The first reason rank reversal can occur is the principal eigenvector approach. The second cause of rank reversal is the relative judgment mode.

The third reason is given by AHP defenders, they attribute rank reversal to 1) the dependence and feedback between alternatives and criteria ; and 2) the scarcities and abundance of alternatives .

3.2 A way to preserve rank

Because the current rank preservation methods have theoretical limitations, and also because the conditions of consistency and independence are not fully considered by the AHP researchers and critics alike, another method is needed to preserve rank under the conditions of consistency and independence when a new alternative is added or when one is deleted. With careful consideration of rank reversal in the AHP, it can be shown that with the use of the weighted geometric mean aggregation rule in place of the arithmetic mean aggregation rule, rank can be preserved when a new alternative is added or when one is deleted. However, the usage of the modified geometric mean aggregation rule to preserve rank must tackle the following two problems: one is that the local priorities which will be synthesized by the modified geometric mean aggregation rule should be obtained through the principal eigenvector approach; the other is that the shortcomings of synthesizing with the geometric mean should be overcome in the aggregation process.

In the next section, we will provide a solution to the two problems presented above.

4. Justification on rank preservation

In this section, we prove that the modified geometric mean aggregation rule can preserve rank via the principal eigenvector approach. Then we prove that the shortcomings of the geometric mean will not occur in the aggregation process.

4.1 The modified geometric mean aggregation rule can preserve rank

With pairwise comparison judgments, the priorities of alternatives are relative and depend on each other. It is reasonable to assume that if all the judgments are consistent and all elements are independent when comparing the alternatives with respect to each criterion, adding or deleting an alternative should preserve the final overall priorities of the alternatives with respect to all the criteria. This is the case when using a weighted geometric mean aggregation rule as will be shown in the proof below. The rank preservation idea can be described as the following theorem:

Theorem 1. In the AHP, when a new alternative is added or when one is deleted, the usage of the weighted geometric mean aggregation rule can guarantee that the proportions of the final weights of the old alternatives remain unchanged if all judgments are consistent and all elements are independent.

Theorem 1 is an enhanced version of rank preservation. In theorem 1, not only the rank of original alternatives can be preserved, but the proportions of the final weights of the original alternatives can also be preserved.

4.2 The shortcomings of synthesizing with the geometric mean can be overcome in the aggregation process

In this section, we prove that the shortcoming of geometric mean can be overcome.

5. Validity check

In this section, two familiar examples are presented which have been widely discussed in the complex arguments regarding rank reversal in the AHP to check the validity of our statement.

6. Discussion and conclusion

AHP critics attribute rank reversal to the principal eigenvector approach and the relative judgment . We disagree.

For the eigenvector approach

As was discussed in section 3.1, many researchers attribute rank reversal to the principal eigenvector approach. Barzilai and Golany , in particular, hold that for all judgments there does not exist any synthesis method which avoids rank reversal. However, the principal eigenvector approach has nothing to do with rank reversal. In pairwise comparison judgments, when all judgments are consistent, the results obtained by the principal eigenvector approach are identical with that by arithmetic mean, geometric mean or logarithmic least square method. Therefore, the principal eigenvector approach is not really the root of rank reversal when all judgments are consistent. In fact, rank preservation is irrelevant to the principal eigenvector approach because it has been proven that rank can be preserved with it when the conditions of consistency and independence are satisfied.

For the relative judgment

In the relative measurement the preference for an alternative is determined by all other alternatives . In this sense the alternatives are not independent from each other for the determination of their priorities. This implies that when one meets relative measurement, dependence and feedback should be considered and hence the *Analytic Network Process* (ANP) should be employed. But if the ANP is introduced into relative measurement, the relative mode of the AHP would disappear, so one could argue just use the ANP. This is an interesting phenomenon because the relative mode is a classification of the AHP, but according to its characteristics it should belong to the category of the ANP.

The reason for this phenomenon is because of the misunderstanding of the relationship between the eigenvector approach and the dependence among alternatives. The eigenvector approach is just a data process method. A number of independent elements should not turn into dependent elements after applying the principal eigenvector approach. Thus, the attribution of rank reversal to the principal eigenvector approach in the relative mode is not correct. Regardless of the absolute judgment or the relative judgment, the weighted geometric mean aggregation rule can preserve rank under the conditions of consistency and independence. Section 4.1 is also a proof for relative judgment, where the weighted geometric mean aggregation rule can preserve rank under the conditions of consistency and independence. This is also true for absolute judgment.

The weighted geometric mean aggregation rule is the solution to Dyer's remarks

Probably the most influential critic on the AHP is Dyer's remarks on rank reversal in *Management Science*. He attributes rank reversal as a symptom of a much more global problem with the AHP: the rankings provided by the methodology are arbitrary. Dyer's methodology arbitrarily uses the eigenvector approach on the scores of the alternatives when the principle of hierarchic composition is assumed. He then points out that the AHP theory does not include any "independence conditions" that can be tested empirically. We disagree because it has been proven that rank can be preserved with the principal eigenvector approach when all judgments are consistent and all elements are independent, wherein the "independence conditions" are considered.

In general, this paper does not question the legitimacy of rank reversal, but rather the rank reversal under the conditions of consistency and independence.

Theoretically, rank preservation should be guaranteed when one meets the conditions. The weighted geometric mean aggregation rule supersedes any other aggregation rules which can avoid rank reversal. In fact, the AHP employs a ratio scale to measure the intensity of preferences of alternatives and criteria, while the weighted geometric mean aggregation rule is also a ratio scale measurement. They are naturally matched. This body of research can augment and expand the AHP theory.

Future research should consider how to address rank reversal in the ANP supermatrix. For example, one could explore what happens when the criteria depend on the alternatives, as well as with tangible and intangible criteria. Such results could strengthen support for the AHP/ANP and its application.

7. References

1. Vargas, L.G., *Comments on Barzilai and Lootsma: Why the Multiplicative AHP Is Invalid: A Practical Counterexample.* Journal of Multi-Criteria Decision Analysis, 1997. **6**(3): p. 169-170.
2. Belton, V. and A.E. Gear, *On a Short-coming of Saaty's Method of Analytic Hierarchies.* Omega-International Journal of Management Science, 1983. **11**(3): p. 513-516.
3. Saaty, T.L., *How to Make a Decision - the Analytic Hierarchy Process.* Interfaces, 1994. **24**(6): p. 19-43.
4. Saaty, T.L., *An Exposition of the AHP in Reply to the Paper "Remarks on the Analytic Hierarchy Process".* Management Science, 1990. **36**(3): p. 259-268.
5. Forman, E.H. and S.I. Gass, *The Analytic Hierarchy Process - An exposition.* Operations Research, 2001. **49**(4): p. 469-486.
6. Barzilai, J. and F.A. Lootsma, *Power Relations and Group Aggregation in the Multiplicative AHP and SMART.* Journal of Multi-Criteria Decision Analysis, 1997. **6**(3): p. 155-165.
7. Barzilai, J., *Deriving weights from pairwise comparison matrices.* Journal of the Operational Research Society, 1997. **48**(12): p. 1226-1232.
8. Barzilai, J. and B. Golany, *Ahp Rank Reversal, Normalization and Aggregation Rules.* Infor, 1994. **32**(2): p. 57-64.
9. Srinivas, K.R., J.E. Aronson, and A. Stam, *SPOT: scheduling programs optimally for television.* Management Science, 1998. **44**(1): p. 83-102.
10. Saaty, T.L., *How to make a decision: The analytic hierarchy process.* European Journal of Operational Research, 1990. **48**(1): p. 9-26.
11. Dyer, J.S., *Remarks on the Analytic Hierarchy Process.* Management Science, 1990. **36**(3): p. 249-258.